# Salt of the Sea. Miscellaneous Notes

Carl Wunsch

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## 1 Halosteric Effects and Sealevel Rise

Munk (2003) noted that the conventional interpretation of the effect of ocean freshening on sealevel change seemed incomplete. His discussion was somewhat terse, and so this is an attempt to do it out with a bit more explanation. Munk's notation is slightly different.

**Salt Perturbation from Fresh Water**

Consider a fluid of depth $h(0)$ of well-mixed density $\rho(0)$ and salinity $S(0)$. It is assumed that

$$\rho(0) = \rho_F (1 + \beta S(0)), \tag{1}$$

with $\rho_F$ being the density of fresh water. The total salt content is

$$\rho(0) S(0) h(0) = \rho_F (1 + \beta S(0)) S(0) h(0) \tag{2}$$

A small amount of fresh water of thickness $\Delta h$, $h(1) = h(0) + \Delta h$, is added to the fluid and then completely mixed again. The total added mass is $\Delta h \rho_F$ where $\rho_F$ is the density of fresh water (the non-linear mixing influence on volume is neglected; see Gille, 2004). The new well-mixed density, $\rho(1) = \rho(0) + \Delta\rho$, is,

$$\begin{aligned} \rho(1)(h(0) + \Delta h) &= \rho(0) h(0) + \rho_F \Delta h \\ &= \rho_F (1 + \beta S(0)) S(0) h(0) + \rho_F \Delta h. \end{aligned} \tag{3}$$

Writing

$$\rho(1) = \rho_F (1 + \beta S(0) + \beta \Delta S),$$

substituting into Eq. (3), neglecting terms of $O(\Delta S \Delta h)$, produces

$$\frac{\Delta S}{S(0)} = -\frac{\Delta h}{h(0)}. \tag{4}$$

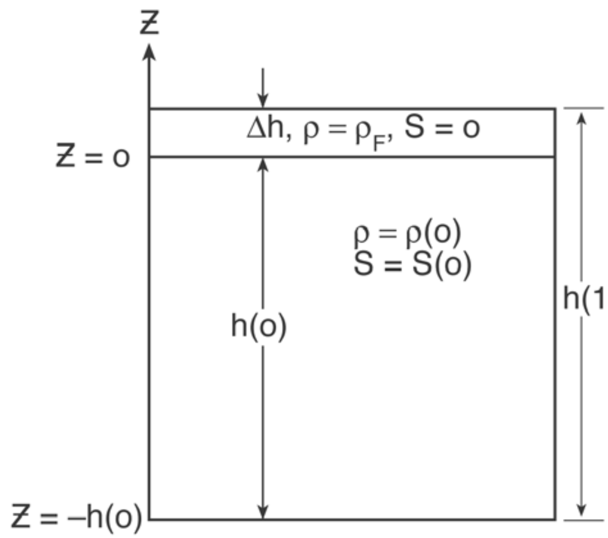


Figure 1: Definition sketch of a fluid of initial depth $h(0)$, salinity $S(0)$ and density $\rho(0)$. A fresh water layer of depth $\Delta h$ is added so that $h(1) = h(0) + \Delta h$.

Alternatively, the total salt content is unchanged, and must be such that,

$$\underbrace{\rho(1) S(1)(h(0) + \Delta h)}_{\text{new salt}} = \underbrace{\rho(0) S(0) h(0)}_{\text{old salt}},$$

or

$$(h(0) + \Delta h)(S(0) + \Delta S)(\rho(0) + \Delta\rho) = \rho(0) S(0) h(0)$$

Expanding

$$h(0) S(0) \varrho(0) + \Delta h S(0) \rho(0) + \Delta S h(0) \rho(0) + \Delta\rho h(0) S(0) + O(\Delta h \Delta S) = h(0) \rho(0) S(0)$$

So that,

$$\Delta S h(0) \rho(0) + \Delta\varrho h(0) S(0) = -\Delta h S(0) \rho(0)$$

But, $\Delta\rho = \rho_F \beta \Delta S$, and thus

$$\begin{aligned}\Delta S &= -\frac{\Delta h S(0) \varrho(0)}{h(0) \varrho(0) + \rho_F \beta h(0) S(0)} \\ &= -\frac{\Delta h}{h(0)} \left( \frac{S(0) \rho(0)}{\rho(0) + \varrho_F \beta S(0)} \right) \approx -\frac{\Delta h}{h(0)} S(0)\end{aligned}$$

That is, again,

$$\frac{\Delta S}{S(0)} = -\frac{\Delta h}{h(0)}.$$

If $h(0) = 4000$m, $\Delta h = 10$cm (about 30-50 years of recent accumulation), $\Delta S/S(0) \approx -2.5 \times 10^{-5}$. Is that measurable?

The new density is $\Delta\rho = \rho_F \beta \Delta S = -\rho_F \beta S(0) \Delta h / h(0) \approx 35 \Delta h / h(0)$.

**Halosteric Change**

What is meant by the steric change? It seems to mean the density shift from changing salinity, somehow converted into a volume or mass change. It's not entirely clear how this is to be interpreted.

Suppose, as above, we add a layer of fresh water, such that $\Delta S = -S(0) \Delta h / h(0)$. The original fluid column with density $\varrho(0)$ generates a pressure at $z = -h(0)$ of,

$$p(0) = -g\rho_F \int_{-h(0)}^{\eta(0)=0} (1 + \beta S(0))\, dz = g\rho_F h(0) (1 + \beta S(0)).$$

The new density, after the fluid is well mixed is,

$$\rho(1) = \varrho_F \left[1 + \beta S(0) \left(1 - \frac{\Delta h}{h(0)}\right)\right],$$

and the new pressure is

$$\begin{aligned} p(1) &= -g\rho_F \int_{-h(0)}^{\eta(1)} \left[1 + \beta S(0) \left(1 - \frac{\Delta h}{h(0)}\right)\right] dz \\ &= g\rho_F (\eta(1) + h(0)) \left[1 + \beta S(0) \left(1 - \frac{\Delta h}{h(0)}\right)\right] \\ &= p(0) + g\rho_F h(0) \beta S(0) \left(-\frac{\Delta h}{h(0)}\right) + g\rho_F \eta(1) (1 + \beta S(0)) + O(\eta \Delta h / h). \end{aligned}$$

If we now suppose $p(1) = p(0)$, as in a level of no motion assumption,

$$h(0) \beta S(0) \frac{\Delta h}{h(0)} = \eta(1) (1 + \beta S(0)).$$

That is,

$$\eta(1) = \frac{\beta S(0) \Delta h}{1 + \beta S(0)} \approx \beta S(0) \Delta h << \Delta h$$

$\eta(1)$ would vanish if the water is fresh $(S(0) = 0)$, as Munk (2003) pointed out.

There seems to be no particular reason to resort to halosteric calculations—one should use the mass/volume change. To convert halosteric changes into sealevel shifts requires additional information (the mass added).

## 2 Salinity Boundary Conditions

Huang (1993) called attention to the use in numerical models of the so-called virtual salt flux boundary condition at the seasurface rather than the actual fresh water flux. Dewar and Huang (1996) worked out several examples showing how different the temporal behavior under the two

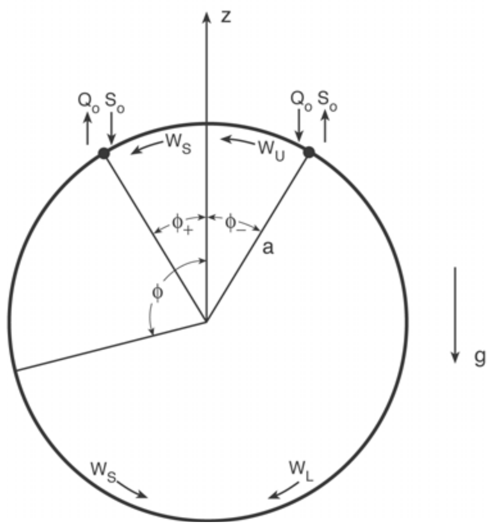


Figure 2: One dimensional loop model, after Wunsch (2005), Yuan and Wunsch (2005). Here $\phi_+ = \phi_-$ and there are either salinity or mass sources configured as shown. $w_S$ is the non-dimensional unidirectional flow driven by salinity sources, $w_U, w_L$ are the two different flows imposed by volume source/sink, with convergence onto the sink.

boundary conditions could be. They did show that the virtual salt flux bc could be recovered from the freshwater flux one if the fresh water forcing were sufficiently weak. Numerical issues aside, Huang's (1993) point is an important one: a boundary source term has been moved from one governing equation to another, and rather general principles suggest that great care needs to be taken to assure that the solutions are truly the same. With the advent in the 1990s of free-surface numerical models, the use of virtual salt flux boundary conditions is regarded as obsolete; nonetheless, a number of models continue to use it.

In an attempt to elucidate the issue, I consider here a simple example, based upon the "loop-Stommel" models discussed by Wunsch (2005), Yuan and Wunsch (2005), with geometry shown in Fig. 2. The flow is assumed one-dimensional with the angle $\phi$ measured clockwise from the top of the loop. At position $\phi_+$ there is a source or sink (variously mass or salt), and at $\phi_-$ a counterbalancing sink or source. Here temperature will be ignored, and the density assumed linear in salinity, $S$ :

$$\rho\left(\phi,t\right)=\rho_f\left(1+\beta S\left(\phi,t\right)\right).$$

### 2.1 Salt Flux

Suppose, initially, and as in Yuan and Wunsch (2005), there is a true salt source, $S_0$, at $\phi_+$ and a "negative" source, $-S_0$, (a sink) at $\phi_-$.

$$\frac{\partial S}{\partial t}+\frac{w}{a}\frac{\partial S}{\partial \phi}-\frac{\kappa}{a^2}\frac{\partial^2 S}{\partial \phi^2}=2\pi S_0[\delta\ (\phi-\phi_+)-\delta\ (\phi-\phi_-)]$$

suppressing the subscript on the diffusion $\kappa$. Here we will consider only $\phi_-=-\phi_+$, the type-3 convection limit. The momentum equation is,

$$\frac{\partial w}{\partial t}=-\frac{\partial p}{a\rho_0 \partial \phi}+g\frac{\rho}{\rho_0}\sin\phi-\epsilon w \tag{5}$$

where the Boussinesq approximation is being used, and $\varepsilon$ is a friction coefficient. In the Boussinesq approximation, the mass flux is really a volume flux. $\rho_0$ might be taken as either $\rho_F$ (the fresh water density) or some higher average value, taking account of the mean salinity. Here we use $\rho_F$. In the Boussinesq approximation,

$$\frac{\partial w}{\partial \phi}=0 \tag{6}$$

Eq. (5) is integrated between $-\pi$ and $\pi$, thus eliminating the pressure:

$$\frac{\partial w}{\partial t}=\frac{g\beta}{2\pi}\int_{-\pi}^{\pi} S\left(\phi,t\right)\sin\phi d\phi-\epsilon w$$

With minor variations, we follow the non-dimensionalization of Yuan and Wunsch (2005), but using salinity rather than temperature as the overall scale. Thus,

$$S=\frac{\alpha^2}{\kappa}S_0S',\quad w=\frac{\kappa}{a}w',\quad t=\frac{a^2}{\kappa}t',$$

and the governing equations become,

$$\frac{\partial S}{\partial t}+w\frac{\partial S}{\partial \phi}-\frac{\partial^2 S}{\partial \phi^2}=2\pi[\delta\ (\varphi-\varphi_+)-\delta\ (\phi-\phi_-)] \tag{7}$$

$$\frac{1}{P_r}\frac{\partial w}{\partial t}=\frac{R_a}{2\pi}\int_{-\pi}^{\pi} S\ (\phi)\sin\phi d\phi-w \tag{8}$$

$$\frac{\partial w}{\partial \phi}=0 \tag{9}$$

with

$$R_a=\sqrt{\frac{a^{\frac{3}{2}}g^{\frac{1}{2}}\triangle_T T_0}{\epsilon\kappa}},\ P_r=a^2\epsilon/\kappa..$$

The primes have been dropped from the nondimensional variables. $R_a$ is a Rayleigh number, $P_r$ a Prandtl number. Eq. (9) was used in integrating Eq. (8) around the loop. The salinity source/sink are the analogue of the use of virtual salinity boundary conditions in general circulation models.

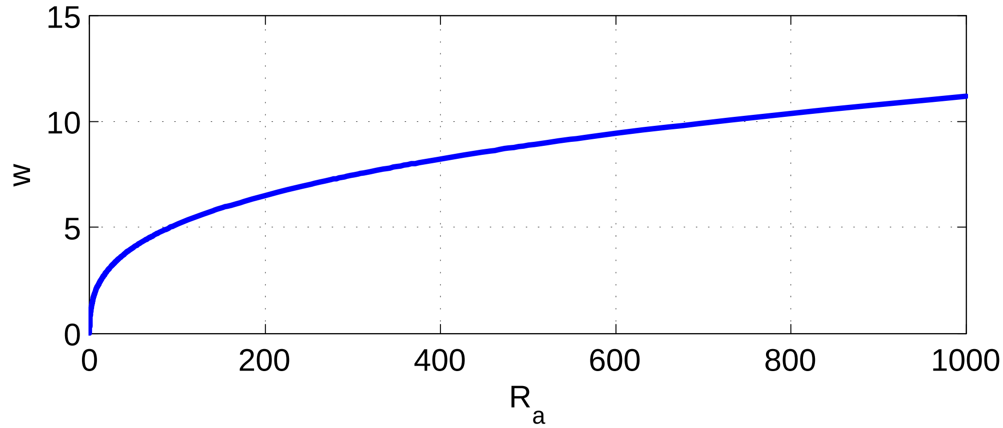


Figure 3: Non-dimensional $w$ plotted versus the Rayleigh number with $\phi_{\pm} = \pm\pi/4$ for salinity source forcing. Flow is clockwise driven by the imposed density difference through the salt flux.

For maximum simplicity, we will only discuss the steady-state, so that:

$$w\frac{\partial S}{\partial \phi} - \frac{\partial^2 S}{\partial \phi^2} = 2\pi[\delta\ (\varphi - \varphi_+) - \delta\ (\phi - \phi_-)] \tag{10}$$

$$w = \frac{R_a}{2\pi}\int_{-\pi}^{\pi} S\ (\phi)\sin\phi d\phi, \tag{11}$$

where $w$ is constant in $\phi$. One proceeds as in Wunsch (2004), expanding $S(\phi)$ and the delta functions in a Fourier series. The non-dimensional velocity satisfies,

$$\left(1 + w^2\right) w - 2R_a \sin\phi_+ = 0, \tag{12}$$

taking $\phi_+ = \pi/4$. $w$ is plotted as a function of $R_a$ in Fig. 3. (The cubic has one real root.) The flow is then clockwise, with a strength that varies with $R_a$. Note that the dimensional $w$ is proportional to $\kappa$, and thus vanishes in the limit $\kappa \to 0$ even with the Rayleigh number going to infinity (see the discussion in Wunsch, 2005, who used a different scaling, and an inverse Rayleigh number). The distribution of salinity in the loop is a strong function of $R_a$, varying from nearly homogeneous, with boundary layers near the source/sink to stably stratified over one-half the loop.

The salt distribution for a particular set of values of $R, w$ is shown in Fig. 4, including two pseudo basins intended to mimic Stommel's (1961) two box model. The combined diffusion and downwelling in the region below the salinity source leads to a near uniform high salinity value, while in the region below the salinity sink there is a competition between upwelling and downward diffusion producing the familiar near-exponential "abyssal recipes" balance.

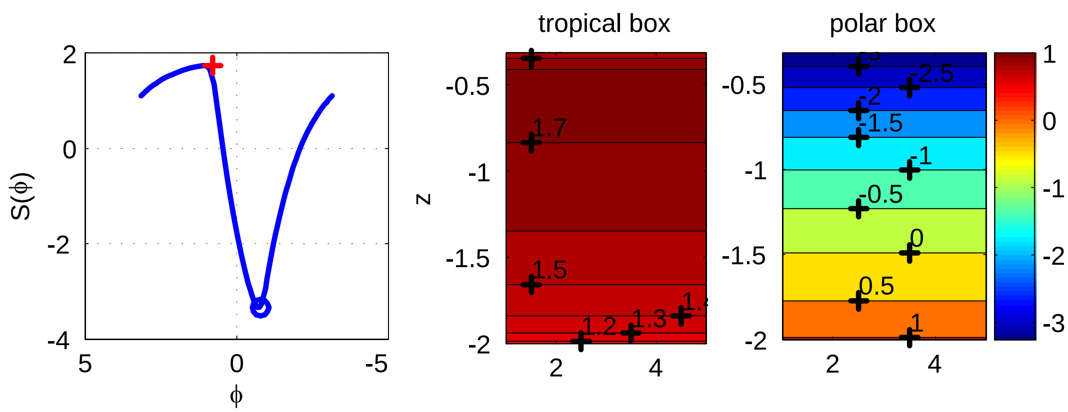


Figure 4: Salinity distribution for $R = 1$, $w = 0.834$ plotted as a function of the angle $\phi$ (left panel), and opened out in two pseudo oceanographic boxes. The "tropical box" (center panel) lies below the salt source and the "polar box" lies below the salt sink. The pseudo depth is $z = -(1 - \cos\phi)$.

## 2.2 Freshwater Flux

Now suppose we formulate the problem in the natural sense in which the salt source corresponds to evaporation (removal of fresh water) and the negative salt flux (out of the tube) corresponds to a supply of fresh water. There is no salinity source because the atmosphere does not carry salt. The dimensional salinity equation becomes

$$\frac{\partial S}{\partial t} + \frac{w}{a}\frac{\partial S}{\partial \phi} - \frac{\kappa}{a^2}\frac{\partial^2 S}{\partial \phi^2} = 0.$$

The momentum equation is still Eq. 8.. In the Boussinesq approximation Eq. (6) becomes,

$$\frac{\partial w}{a\partial \phi} = -2\pi Q_0[\delta\ (\varphi - \varphi_+) - \delta\ (\phi - \phi_-)] \tag{13}$$

that is, with s source and sink. Note the minus sign. Volume is now being removed at $\phi = \phi_+$, (net evaporation) the nominal tropics, and added at $\phi = \phi_-$ (net precipitation) but overall volume is fixed. To retain the previous scaling, we will take $S_0 = Q_0$, and then, in the non-dimensional steady-state,

$$w\frac{\partial S}{\partial \phi} - \frac{\partial^2 S}{\partial \phi^2} = 0 \tag{14}$$

$$\int_{-\pi}^{\pi} w\left(\phi\right) d\phi = R_a \int_{-\pi}^{\pi} S\ \left(\phi\right) \sin\phi d\phi \tag{15}$$

$$\frac{\partial w}{\partial \phi} = -2\pi \bar{Q}_0 [\delta\ \left(\varphi - \varphi_+\right) - \delta\ \left(\phi - \phi_-\right)], \quad \bar{Q}_0 = Q_0 \frac{a^2}{\kappa}. \tag{16}$$

$w$ is now a function of $\phi$. Coupling between the volume flux and salinity occurs through Eq. (15). Eq. (16) can be integrated directly so that,

$$\begin{aligned} w &= \beta_0 + w'\left(\phi\right) \\ w'\left(\phi\right) &= 0, \quad \phi_- \leq \phi \leq \phi_+ \\ &= -2\pi \bar{Q}_0, \quad \phi_+ \leq \phi \leq 2\pi + \phi_-, \end{aligned} \tag{17}$$

where $\beta_0$ is an unknown constant. The solution to Eq. (14) is $S = \bar{S}$, a constant, and Eq. (15) implies

$$\int_{-\pi}^{\pi} w\left(\phi\right) d\phi = 0$$

determining $\beta_0$. Then the final solution is

$$\begin{aligned} w\left(\phi\right) &= w_U = -2\bar{Q}_0 \left(\pi - \phi_+\right), \quad -\phi_+ \leq \phi \leq \phi_+ \\ &= w_L = 2\bar{Q}_0 \phi_+, \quad \phi_+ \leq \phi \leq 2\pi - \phi_+ \\ S &= \bar{S} \end{aligned}$$

shown in Fig.2) which is very different from the solution using virtual salt fluxes. That is, the flow is away from the volume source, and toward the sink, both above and below the forcing. This result contrasts greatly with the salinity-forced system, where the flow is uniform of one-sign, counterclockwise everywhere. The salinity distributions are also very different. Even in this very simple geometry, the two boundary conditions produce radically different steady states. Deducing what happens in a GCM is not so easy. The inference is that one should use the physical, volume flux, boundary condition. Interpreting GCM solutions with virtual salt flux boundary conditions, particularly where strong fresh water forcing has been imposed, is very difficult. The loop, in the guise of the Stommel (1961) model, has been widely used to make inferences about GCM behavior under climate change. In addition to the issue of geometry taken up by Wunsch (2004), there is a fundamental problem of sensitivity to the form of the boundary conditions imposed. (This conclusion simply reinforces those of Huang, 1993; Dewar and Huang, 1996.)

Note that when forming or melting seaice, there can be a true salt flux (ice has a finite salinity). One must then use a combination of volume and salt flux boundary conditions in those regions. Much of the GCM discussion of the salt boundary condition (e.g., Griffies, 2001) involves the movement of the free surface. The loop model here does not have any counterpart to that—the different flows occuring simply because of the shift in the equation which is being driven by the sources.

# 3 Sampling Problems

Antonov et al. (2002) infer, from observed hydrography, a freshening of the ocean over 50 years of a halosteric equivalent of 0.055cm/y or about 2.75 cm of water. The implied salinity change is then $\Delta S/S\,(0) = -\Delta h/h\,(0) \approx -3\times10^{-2}\text{m}/4000\text{m}= -7.5\times10^{-6}$. The question raised here is whether such an inference is justified, given the oceanic sampling problems? To suggest the issue, Fig. 5 shows the distribution of all data in temperature and salinity at 300m in the Levitus et al. (2001) compilation. Although in recent years, coverage in the Southern hemisphere has improved considerably by comparison, the enormous holes, particularly in salinity, over much of the world ocean call into question whether the time mean salinity, much less its temporal changes, could conceivably be accurately calculated? It is that question we seek to explore.

Consider the problem of making an estimate of the mean value of a field. The observations are defined as $y_i = m + n_i$ where $m$ is the true salinity mean and $n_i$ is the noise in the measurement. Make the strong assumption that the noise is unbiassed, $\langle n_i \rangle = 0$, and that something is known of the error covariance, $\mathbf{R} = \langle n_i n_j \rangle$. Make no prior assumption about $m = \langle S_i' \rangle$. One seeks its best estimate (e.g., Eq. 3.6.29 of Wunsch, 1996, with $m_0^2 \to \infty$), the best estimate which is,

$$\tilde{m} = \frac{1}{\mathbf{D}^T\mathbf{R}_{nn}^{-1}\mathbf{D}}\mathbf{D}^T\mathbf{R}_{nn}^{-1}\mathbf{y}$$

with uncertainty,

$$\mathbf{P} = \left\langle (\tilde{m} - m)^2 \right\rangle = \frac{1}{\mathbf{D}^T\mathbf{R}_{nn}^{-1}\mathbf{D}}.$$

Here $\mathbf{D}$ is a column vector of ones, $\mathbf{D} = [1, 1, ...1]^T$ corresponding to each of the measurements.

In this formulation, the signal is the mean salinity, and thus $\mathbf{n}$ includes both errors of measurement and any spatial structure in $S_i$ that makes it locally different from its global mean value. The main problem then is to estimate $\mathbf{R}_{nn}$, which as written, includes horizontally, vertically, and temporally displaced measurements.

One might simplify this problem by assuming that all measurements made within five years of each other are treated as simultaneous, removing vertical singular vectors to suppress vertical covariances, and by guessing at a horizontal covariance at fixed levels. Is it worth doing?

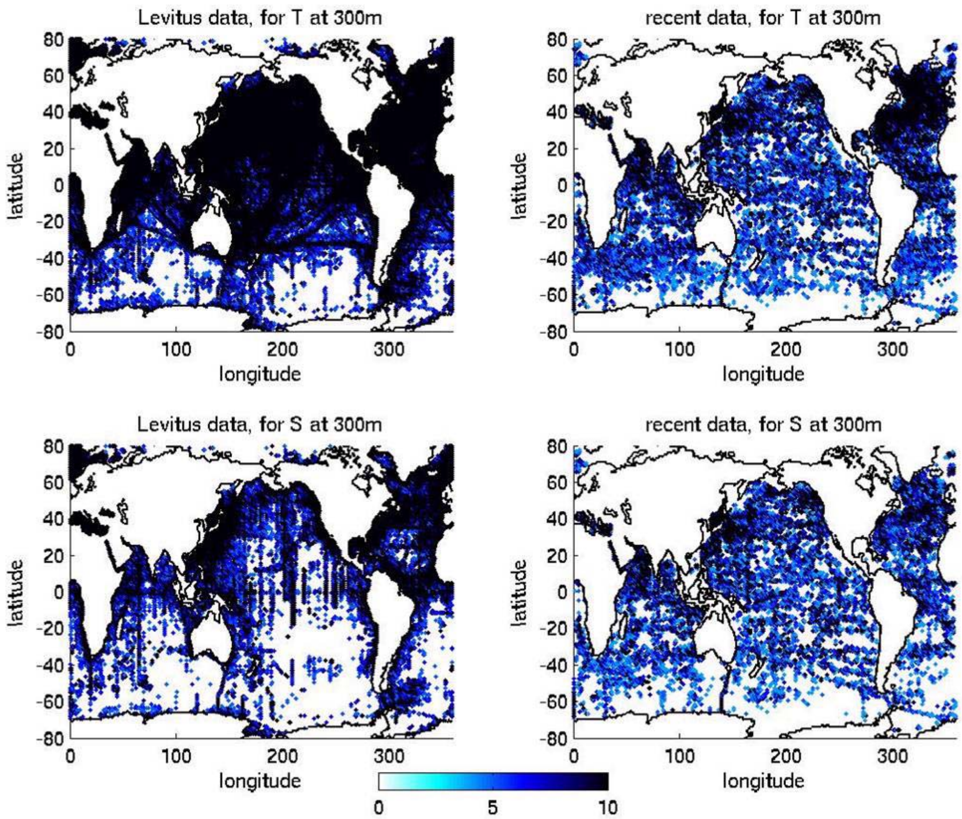


Figure 5: Left column shows the positions where the World Ocean Atlas has more than four data points in a 1 degree square for temperature and salinity at 300m. Right column shows data that became available more recently. Coverage declines rapidly with depth.

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