Deep-Sea Research, 1968, Vol. 15, pp. 251 to 258. Pergamon Press. Printed in Great Britain.

On the propagation of internal waves up a slope*

CARL WUNSCH†

(*Received* 15 *January* 1968)

Abstract—Simple analytic solutions exist for the problem of periodic internal waves in an ocean of constant Brunt–Vaisala frequency where the bottom slope is linear. The solutions indicate a linear decrease of wavelength, and an increase in amplitude of the velocity field as the intersection of the bottom and the surface is approached. A logarithmic singularity exists at the intersection indicating high dissipation or breaking. The slope itself is a region of high shear.

INTRODUCTION

IN DESIGNING a series of measurements of internal waves with an antenna-like array, it is clear that adjustments must be made to the measurements for the distorting effects of topography. The few pertinent measurements that have been made, e.g., HAURWITZ, STOMMEL and MUNK (1959) and PARKER and WUNSCH (1967) indicate that internal waves propagating up the Bermuda slope are distorted, and in a sense amplified by the rapid change in oceanic depth.

The Haurwitz, Stommel and Munk result showed a marked lack of coherence, at all frequencies, between thermistors placed on the bottom a distance of 1 mile apart in the horizontal, and 450 m in the vertical. This lack of coherence has been ascribed by WUNSCH and PARKER (1967) to nonlinear processes degrading the wave trains as they move up the slope.

In this note we investigate, as a first step toward a theory of wave-topography interaction, the changes in a sinusoidal internal wave train as it encounters a shoaling region. A linear theory is used.

EQUATIONS OF MOTION

Currently feasible internal wave antennas are capable of defining internal waves of rather short wavelength, and we will confine this study primarily to the high-frequency, short wavelength, and hence non-rotating case. Rotation can be easily introduced. We assume two-dimensional motion in a stably stratified Boussinesq fluid. The appropriate perturbation equations are (LAMB, 1932, p. 378):

$$\rho_0 u_t = -p'_x \tag{1}$$

$$\rho_0 w_t = -p'_z - g\rho' \tag{2}$$

$$\rho_t' + w\,\rho_{0z} = 0 \tag{3}$$

$$u_x + w_z = 0. \tag{4}$$

*Contribution No. 2070 from the Woods Hole Oceanographic Institution.
†Department of Geology and Geophysics, Massachusetts Institute of Technology

The velocities u, w are positive to the right and vertically upward respectively. ρ_0 is the mean (stable) density field, and is taken to be constant except in equation (3). ρ' and p are the perturbation density and pressure fields.

Introducing a stream function $u = -\hat{\psi}_z$, $w = \hat{\psi}_x$ we obtain, from (1) to (4),

$$\nabla^2 \hat{\psi}_{tt} + N^2 \hat{\psi}_{xx} = 0 \tag{5}$$

where

$$N = \left(\frac{-g}{\rho_0}\frac{\partial \rho_0}{\partial z}\right)^{\frac{1}{2}}$$

is the Brunt–Vaisala frequency.

If we assume periodic motion, such that,

$$\hat{\psi} = \psi\, e^{-i\sigma t}$$

then (5) becomes

$$\psi_{zz} - \frac{1}{c^2}\psi_{xx} = 0, \quad c^2 = \frac{\sigma^2}{N^2 - \sigma^2}. \tag{6}$$

If $c^2 > 0$, this equation is hyperbolic in the space coordinates with characteristics $x \pm cz =$ constant.

THE BEACH PROBLEM

We will consider a periodic internal wave disturbance propagating from an "abyssal" region with a flat bottom, onto a "beach" of constant slope. Let the line $z = -\gamma x$, $z \geqslant -z_1$ $z = -z_1$, $x \geqslant x_0$ be the ocean bottom. For further simplicity we will assume the upper boundary ($z = 0$) is rigid (see Fig. 1), making the beach into a wedge.

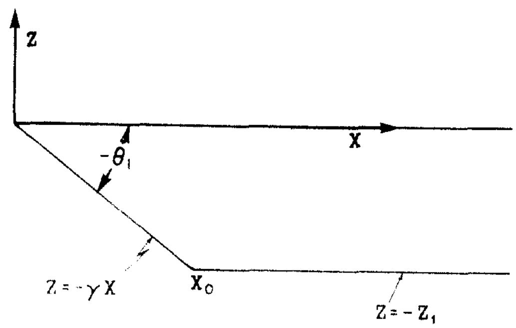


Fig. 1. Beach and abyssal regions of the model ocean.

The boundary condition is that $\psi = 0$ on the boundary. We will only consider the case $N =$ constant. Under this condition the general solution to (6) is

$$\psi = F(cx - z) + G(cx + z) \tag{7}$$

where F and G are arbitrary functions. Since (6) is not separable in a wedge, we expect only partial ability to construct a solution there.

In the polar coordinates,

$$x = r\cos\theta, \quad z = r\sin\theta$$

we have

$$\psi = F[r(1+c^2)^{\frac{1}{2}}\cos(\theta+\alpha)] + G[r(1+c^2)^{\frac{1}{2}}\cos(\theta-\alpha)], \quad \alpha = \tan^{-1}\frac{1}{c}. \tag{8}$$

There are two sets of useful solutions

$$F^A = \sin[p\ln(cx-z)], \quad G^A = -\sin[p\ln(cx+z)] \tag{9}$$

and

$$F^B = -\cos[p\ln(cx-z)], \quad G^B = \cos[p\ln(cx+z)] \tag{10}$$

which yield

$$\psi^A = 2\sin\left[\frac{p}{2}\ln\left(\frac{cx-z}{cx+z}\right)\right]\cos\left[\frac{p}{2}\ln(c^2x^2-z^2)\right]$$

$$= 2\sin\left[\frac{p}{2}\ln\left(\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right)\right]\cos\left\{\frac{p}{2}\ln[r^2(1+c^2)]\cos(\theta+\alpha)\cos(\theta-\alpha)\right\}$$

$$\psi^B = 2\sin\left[\frac{p}{2}\ln\left(\frac{cx-z}{cx+z}\right)\right]\sin\left[\frac{p}{2}\ln(c^2x^2-z^2)\right]$$

$$= 2\sin\left[\frac{p}{2}\ln\left(\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right)\right]\sin\left\{\frac{p}{2}\ln[r^2(1+c^2)]\cos(\theta+\alpha)\cos(\theta-\alpha)\right\} \tag{11}$$

which we will refer to as the A and B solutions respectively.

If the angle $-\theta_1$, corresponds to the line $z = -\gamma x$ then choosing

$$\frac{p}{2} = \frac{n\pi}{\ln\left(\frac{c+\gamma}{c-\gamma}\right)} = \frac{n\pi}{\ln\left[\frac{\cos(-\theta_1+\alpha}{\cos(\theta_1+\alpha)}\right]}$$

we have

$$\psi_n{}^A = 2\sin\left[\frac{n\pi}{\ln\left(\frac{c+\gamma}{c-\gamma}\right)}\ln\left(\frac{cx+z}{cx-z}\right)\right]\cos\left[\frac{n\pi}{\ln\left(\frac{c+\gamma}{c-\gamma}\right)}\ln(c^2x^2-z^2)\right]$$

$$= 2\sin\left[\frac{n\pi}{\ln\left(\frac{\cos(-\theta_1+\alpha)}{\cos(\theta_1+\alpha)}\right)}\ln\left(\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right)\right]$$

$$\cos\left[\frac{n\pi}{\ln\left(\frac{\cos(-\theta_1+\alpha)}{\cos(\theta_1+\alpha)}\right)}\ln\{r^2(1+c^2)\cos(\theta-\alpha)\cos(\theta+\alpha)\}\right]. \tag{12}$$

$$\psi_n{}^B = 2\sin\left[\frac{n\pi}{\ln\left(\frac{c+\gamma}{c-\gamma}\right)}\ln\left(\frac{cx-z}{cx+z}\right)\right]\sin\left[\frac{n\pi}{\ln\left(\frac{c+\gamma}{c-\gamma}\right)}\ln(c^2x^2-z^2)\right]$$

$$= 2\sin\left[\frac{n\pi}{\ln\left(\frac{\cos(-\theta_1+\alpha)}{\cos(\theta_1+\alpha)}\right)}\ln\left(\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right)\right]$$

$$\sin\left[\frac{n\pi}{\ln\left(\frac{\cos(-\theta_1+\alpha)}{\cos(\theta_1+\alpha)}\right)}\ln(r^2(1+c^2)\cos(\theta-\alpha)\cos(\theta+\alpha)\right] \tag{13}$$

which clearly vanish along the lines $z = 0$ ($\theta = 0$) and $z = -\gamma x$ ($\theta = -\theta_1$).

There is no line $r =$ constant such that $\psi = 0$ there; thus only a partially separated solution has been found, as was expected.

The solutions are self-similar in r, i.e., if we rescale r by a constant k, $\hat{r} = kr$, the only change in the solutions is a phase change in the trigonometric functions. This is another way of saying that a wedge has no natural length scale. The phase must be determined from additional information.

If we put $s = p \ln (cx - z) - \sigma t$ we can define a local wave number vector for the F^A solutions

$$(k_x, k_z) = \left(\frac{\partial s}{\partial x} \cdot \frac{\partial s}{\partial z}\right)$$
$$= \left(\frac{pc}{cx - z}, \; -\frac{p}{cx - z}\right)$$

and similarly for the other solutions. The wave lengths go to zero as the apex of the wedge is approached. Since

$$F^A = \sin [p \ln (cx - z)]$$
$$= \frac{1}{2i} (e^{ip \ln (cx - z)} - e^{-ip \ln (cx - z)})$$

the F^A solution is a standing pattern of waves propagating with wave number vector perpendicular to the characteristic lines $z - cx =$ constant, with phase velocity

$$V_p = \left(\frac{\sigma (cx - z)}{-pc}, \; \frac{\sigma (cx - z)}{p}\right).$$

The A solutions give a velocity field of

$$u^A (x, z) = - \psi_z{}^A (x, z) = \frac{p}{cx - z} \cos [p \ln (cx - z)] + \frac{p}{cx + z} \cos [p \ln (cx + z)]$$

$$w^A (x, z) = \psi_x{}^A (x, z) = \frac{pc}{cx - z} \cos [p \ln (cx - z)] - \frac{pc}{cx + z} \cos [p \ln (cx + z)]$$

which have a singularity at the apex. The increase in amplitude of the velocity field is proportional to $1/r$ as the apex is approached, a result of the linear decrease in wavelength of the stream function.

The amplitude of the vertical velocity w increases as $1/(cx + z)$ as the line $z = - cx$ is approached. This is consistent with the qualitative conclusion of Parker and

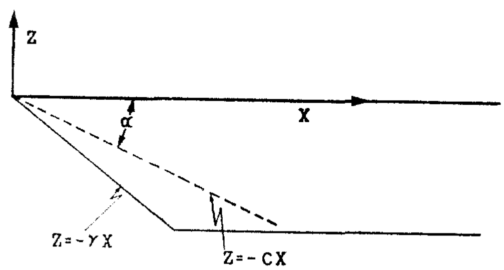
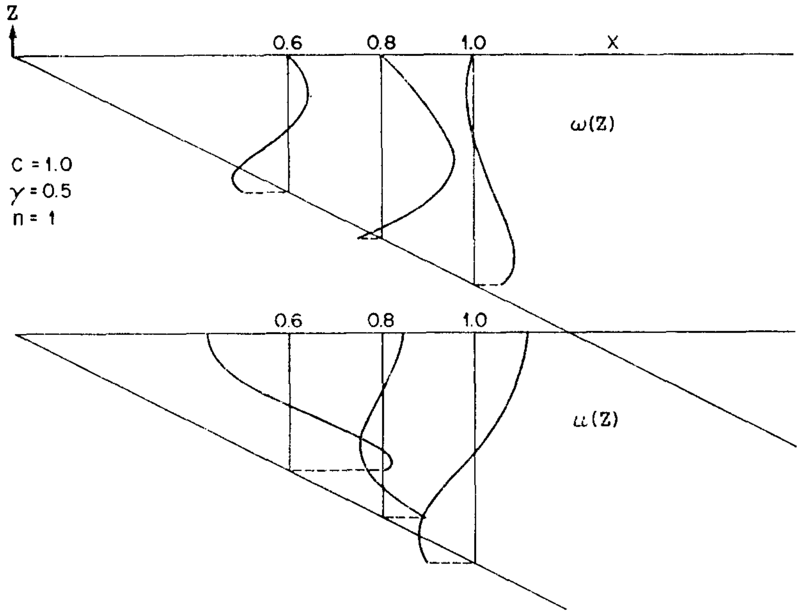


Fig. 2. The critical line $z = - cx$ for which solutions are singular.



Figs. 3. Vertical and horizontal velocities at a number of vertical sections for modes 1 and 2, of the A solutions.

Wunsch that temperature oscillations are strongest on the bottom, where there is a slope.

Some representative illustrations of the velocity field for the A solutions are shown in Figs. 3–6. In Figs. 3, 4, and 6, $c = 1{\cdot}0$. The intensification into a "nose" along the

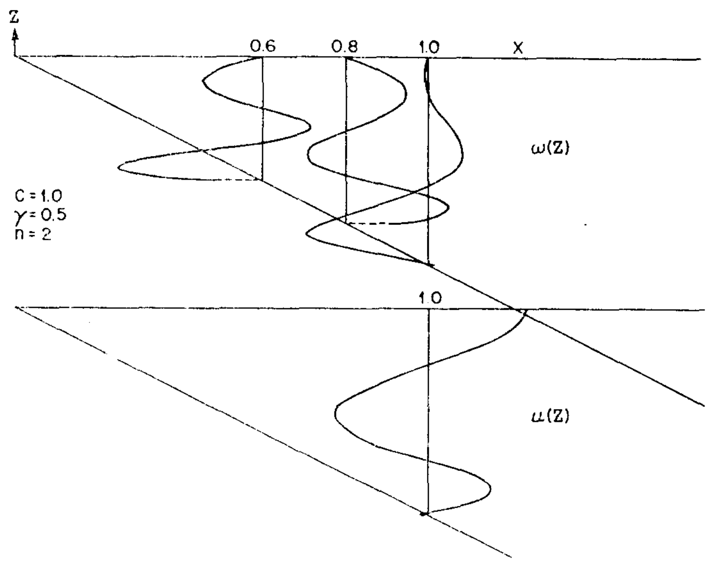


Fig. 4. Vertical and horizontal velocities at a number of vertical sections for modes 1 and 2, of the A solutions.

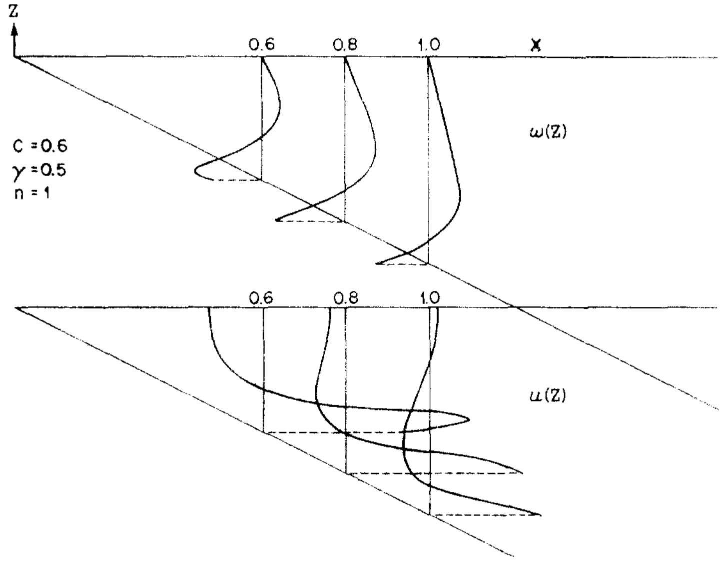


Fig. 5. Vertical and horizontal velocities at a number of vertical sections for modes 1 and 2, of the *A* solutions.

slope is clear, particularly in Fig. 5 where $c = 0{\cdot}6$ and the slope is near the critical angle. Figure 6 is a section at constant depth $z = -1{\cdot}0$, intersecting the slope. The steepening of the crests and shortening of the wavelength is very apparent. The decrease in amplitude for small x is the result of approaching the slope.

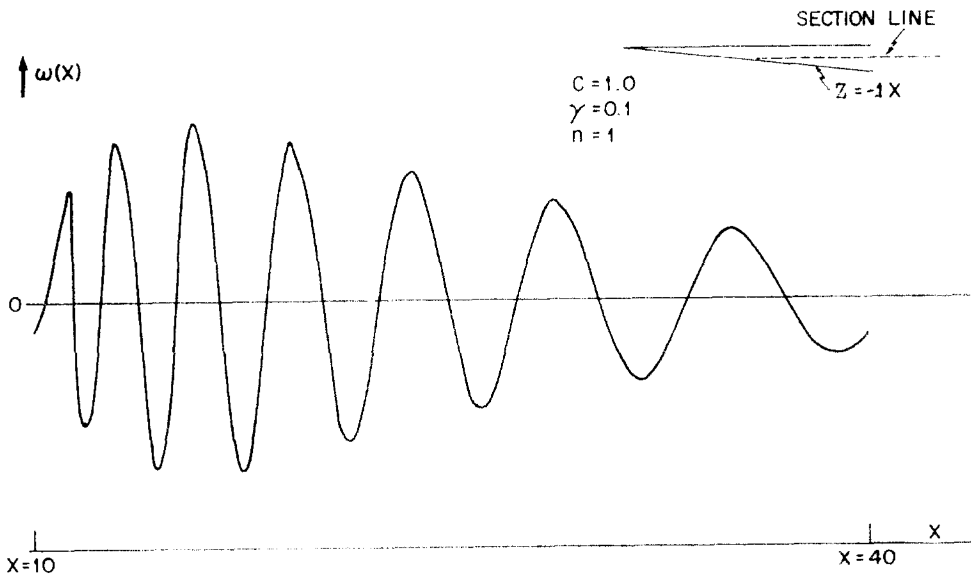


Fig. 6. Vertical velocity through a horizontal plane for mode 1 of *A* solutions.

The energy density for these standing modes is

$$E = \rho_0 \left\{ \frac{u^2 + w^2}{2} + \frac{N^2}{\sigma^2} w^2 \right\}$$
$$= \frac{\rho_0}{2} \left\{ \psi_z^2 + \left(1 + \frac{2N^2}{\sigma^2}\right) \psi_x^2 \right\}$$

which for the A mode is

$$E = \frac{p^2 \rho_0}{2} \left\{ \frac{\cos^2 [p \ln (cx - z)]}{(cx - z)^2} \left[1 + c^2 \left(1 + \frac{2N^2}{\sigma^2}\right)\right] + \frac{\cos^2 [p \ln (cx + z)]}{(cx + z)^2} \left(1 + c^2 \left(1 + \frac{2N^2}{\sigma^2}\right)\right)\right) + 2 \frac{\cos [p \ln (cx - z)] \cos [p \ln (cx + z)]}{c^2 x^2 - z^2} \left[1 - c^2 \left(1 + \frac{2N^2}{\sigma^2}\right)\right].$$

Thus along any line $z = -sx$, the energy density increases as $1/x^2$ as $x \to 0$. The nature of the singularity at the origin in velocity and energy is similar to that found in the classical problem of a surface wave propagating up a beach (STOKER, 1957).

If the wedge angle $-\theta_1$, exceeds the critical angle

$$-\alpha = -\tan^{-1} \frac{1}{c}$$

then $z + cx$ will vanish within the wedge, and the solution will be singular. This line corresponds to a characteristic running directly to the apex of the wedge (Fig. 2). The solution is mathematically permissable, since weak discontinuities are expected along characteristics when singularities occur in the boundary conditions (in this case, at the corner). Physically, this solution corresponds to an infinite shear and is an artifice of the inviscid assumption.

In any case, the slope itself is a region of high shear, which increases as the slope angle approaches the critical angle. Whether this leads to scouring of slopes, or strong mixing processes in the ocean is the subject of some conjecture.

To finish the problem, it is necessary to match the beach region to the abyssal region. In principle, we can go through a Gram–Schmidt process and construct 2 orthonormal sequences $\phi_n{}^A$ and $\phi_n{}^B$ from the sequences $\psi_n{}^A$ and $\psi_n{}^B$. Alternatively, we can inquire into the nature of the abyssal solution necessary to excite a single one of the beach modes. The general solution in the abyssal region is

$$\psi^{\text{II}} = \sum_{n=1}^{\infty} \sin \frac{n\pi z}{z_1} \left(a_n \sin \frac{n\pi c}{z_1} x + b_n \cos \frac{n\pi c}{z_1} x \right).$$

We can match this to a beach mode, say $\psi_1{}^A$ at the line $x = x_0$. Since the wedge solutions are standing modes, as much energy is reflected as is incident. We thus expect standing oscillations in the abyssal region. We have then that

$$\psi_1{}^A (x_0, z) = \sin [p \ln (cx_0 - z)] - \sin [p \ln (cx_0 + z)] = \psi^{\text{II}} (x_0, z)$$
$$\frac{\partial \psi_1{}^A}{\partial x} (x_0, z) = \frac{pc}{cx_0 - z} [\cos [p \ln (cx_0 - z) - \cos [p \ln (cx_0 + z)] = \frac{\partial \psi^{\text{II}}}{\partial x} (x_0, z)$$
$$p = \frac{2\pi}{\ln \left(\frac{c + \gamma}{c - \gamma}\right)}.$$

Notice that we have imposed a Cauchy boundary condition on the matching line. This leads to

$$a_n = H_n \sin \frac{n\pi c}{z_1} x_0 + I_n \frac{z_1}{n\pi c} \cos \frac{n\pi c}{z_1} x_0$$

$$b_n = H_n \cos \frac{n\pi c}{z_1} x_0 - \frac{z_1}{n\pi c} I_n \sin \frac{n\pi c}{z_1} x_0$$

$$H_n = \frac{2}{z_1} \int_{-z_1}^{0} \psi_1{}^A (x_o, z) \sin \frac{n\pi z}{z_1} \, dz \, ; I_n = \frac{2}{z_1} \int_{-z_1}^{0} \frac{\partial \psi_1{}^A}{\partial x} \sin \frac{n\pi z}{z_1} \, dz.$$

If an incident wave is assumed in the abyssal region, then a sum of A and B modes in the beach region will give an incident wave there, the singularity at $z = x = 0$ acting as an energy sink. Physically, either very rapid dissipation through friction or breaking probably takes place.

DISCUSSION

The hyperbolic character of this system is presumably the result of the approximations that have been made, specifically the assumptions of a steady state, linearity, and a lack of viscosity. Unless considerable care is exercised, the use of a hyperbolic system in a bounded region can lead to anomalous solutions. SANDSTROM (1966) solved this equation by the method of characteristics and found that the radiation flux in certain cases was highly unusual.

In this note we simply wish to point out that there is probably no fundamental objection to using this system if the solutions obtained remain physically realistic. In the analogous case of a homogeneous rotating fluid treated by GREENSPAN (1964, 1965) the dynamics are governed by a hyperbolic equation. Greenspan shows that viscous boundary layer corrections remove all mathematical and physical difficulties. For the problem considered in the last section, a similar correction is probably necessary though the details will not be pursued here.

Acknowledgements—I would like to thank Dr. J. B. KELLER for several patient discussions of hyperbolic systems. Dr. M. RATTRAY and Dr. C. ROOTH helped talk the problem through. This work was supported by the Office of Naval Research under Contract Nonr-3963(31) with the Massachusetts Institute of Technology.

REFERENCES

GREENSPAN H. (1964) On the transient motion of a contained rotating fluid. *J. fluid Mech.*, **20**, 673–696.

GREENSPAN H. (1965) On the general theory of contained rotating fluid motions. *J. fluid Mech.*, **22**, 449–462.

HAURWITZ B., H. STOMMEL and W. MUNK (1959) On the thermal unrest in the ocean. In: *Rossby Memorial Volume*, Rockefeller Institute Press, N.Y., 74–94.

LAMB H. (1932) *Hydrodynamics*. Cambridge University Press, 738 pp.

PARKER C. and C. WUNSCH (1967) Observations of thermal fluctuations on the Bermuda slope. Unpublished manuscript.

SANDSTROM H. (1966) The importance of topography in generation and propagation of internal waves. Ph.D. Dissertation, Univ. California, San Diego.

STOKER J. J. (1957) *Water waves*. Interscience, N.Y., 567 pp.

WUNSCH C. and C. PARKER (1967) Upper thermocline internal wave measurements. (Abstract only) *Trans. Am. Geophys. Un.*, **48**, 140.