Tides of Global Ice-Covered Oceans

Preliminary Incomplete Draft

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Abstract

The tides of an ice-covered ocean are examined using a Cartesian representation of the elastic and fluid equations. Although unconstrained by any observations, the ocean tides of a Neoproterezoic "snowball" Earth could be significantly larger than they are today. With the remaining ocean being substantially shallower than today, time-mean tidal-residual circulations could have been set up that are competitive with the circulation driven by geothermal heating. In any realistic configuration, the snowball Earth would have an ice cover that is in the thin shell limit, but by permitting the ice thickness to become large, more interesting ice tidal response can be found, ones conceivably of application to bodies in the outer solar system or hypothetical exoplanets.

1 Introduction

Several reasons exist for an exploration of the tides occurring in ice sheets, whether floating or land-confined. One reason is suggested by evidence that approximately 600 million years ago, during the Neoproterezoic, the entire Earth may have frozen, being everywhere covered with ice. Over the ocean a floating ice sheet may have existed with an estimated of several kilometers (the hard "snowball Earth"). Discussions of the evidence, primarily geological in nature, can be found in Hoffman and Schrag (2002). Ashkenazy et al. (2014; hereafter, A2014), describe a theoretical/modelling study of the oceanic circulation that might exist under an oceanic ice cover of order of several kilometers. The forcing they assume is purely geothermal, at the modern rate of roughly 0.1W/m^2 (Pollack et al., 1993; Davies, 2013), with some localized maxima over ridge-crests. They find an equatorially enhanced meridional overturning circulation, with transports up to 30Sv with a nearly homogeneous ocean, both in temperature and salinity. Some account is taken of the oceanic interaction with the overlying ice sheet.

Whether or not a snowball Earth actually existed, the question of what the ocean might be like under such circumstances is an interesting theoretical problem. A modern analogue is in the outer solar system satellites Enceladus and Europa which, also hypothetically, have fluid oceans covered by multi-kilometer thick ice sheets. In contrast to the A2014, solution, discussion of behavior of those oceans has centered on tidal forcing (Greenberg et al., 1998;...).

A comparatively large literature exists on tides induced in ice sheets by the oceanic tidal forcing at the outflow (e.g., Thomas, 2007; Arbic et al., 2008). In contrast, a tangential calculation here is the body tide induced directly in very large ice sheets far from oceanic influence, and the tides induced in the ocean when overlain by an effectively infinitely thick ice cover. Some of the parameter ranges used here are far beyond anything reasonable for the Earth. Perhaps they have some relevance for another planet or satellite.

Dynamical discussion of continental scale ice sheets such as those in Antarctica and Greenland is difficult for a number of reasons, including the specification of the appropriate boundary conditions at the inaccessible bottom of the glacier. One can speculate that the tidal response observable at the glacier surface is sufficiently sensitive to the bottom boundary conditions that those conditions might be inferred.

2 A Cartesian Configuration

Because of all of the uncertainties of the physical setting of the Neoproterezoic Earth, the goal here is to understand the basic physics and to find orders of magnitude of the effects. Only a two-dimensional Cartesian system as in the Airy "canal theory" of water tides (Lamb, 1932), is used. Consider the situation in Fig. 1, in which an ice sheet of uniform thickness $\bar{d}$ overlies an ocean of constant depth d. Below the ocean is an infinite elastic half-space. The fluid motion is computed with the half-space not moving, as if with the ocean tide being computed relative to the sea floor. Conceptually, as with ocean tides measured from tide gauges, tides within the elastic half-space will produce a modified tidal potential, $U = U_0 \left(1 + k_L - h_L\right)$, where k_L, h_L are the conventional Love numbers (Munk and MacDonald, 1960; Lambeck, 1988). The net tide generating potential will be assumed to be,

$$U = gHe^{ikx - i\sigma t} = g\eta_{Eq}, \tag{1}$$

so that the fluid equilibrium height would be $|\eta_{Eq}| = H$, but with the half-space subsequently treated as completely rigid (unmoving).

2.1 Equations of an Elastic Sheet

Rheological properties of ice, whether on land or floating, are not simple—encompassing elastic, viscous, and plastic flow laws. MacAyeal and Sergienko (2013) propose that for time-scales of less than about 10 days, treating sea ice as elastic is appropriate and thus reasonable for describing ordinary semi-diurnal or diurnal tides. Discussion of long-period tides, or the evection-dominated ones in outer-solar system satellites, requires revisiting the question.

The ice is treated here as purely elastic with Lamé constants λ, μ and the physical and mathematical structure of the problem the free-mode analysis of Bromwich (1898), Press and Ewing (1951), and Ewing, Jardetzky and Press (1957, Ch. 5), but in the presence of a body-force. The Cartesian system governing an elastic plate is, {elastic1}

$$\bar{\rho}\frac{\partial^2 \bar{u}}{\partial t^2} = -\sigma^2 \bar{\rho}\bar{u} = (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z}\right) + \mu \nabla^2 \bar{u} + \bar{\rho}\frac{\partial U_T}{\partial x} \tag{2a}$$ {elasta}

$$\bar{\rho}\frac{\partial^2 \bar{w}}{\partial t^2} = -\sigma^2 \bar{\rho}\bar{w} = (\lambda + \mu)\frac{\partial}{\partial z}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z}\right) + \mu \nabla^2 \bar{w} - g\bar{\rho} \tag{2b}$$ {elastb}

Variables $\bar{u}, \bar{w}$ in the plate are *displacements,* not velocities. Barred variables, $\bar{u}$, etc., will refer to displacements in the ice layer, unbarred ones to corresponding *velocities* in the ocean. $\bar{\rho}$ is the density of ice, U has no vertical dependence in the thin ice layer and no $y-$dependence is considered. The background gravity g produces a resting static pressure $\bar{d} - \bar{\rho} g z$ in the ice.

What follows is in the spirit of the paper by Bromwich (1898) and who, as done in similar problems by Rayleigh and Love, defined the pressure as,

$$\bar{p}(x, z) = -\lambda\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z}\right), \tag{3}$$

taken as finite, but otherwise treated the medium as incompressible with,

$$\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z}\right) = 0, \tag{4}$$ {nondiverge}

and implying $\lambda \to \infty$. One advantage of this system is that it increases the resemblance between the elastic and fluid equations. The sign of $\bar{p}$ has been reversed here from the Bromwich definition, conventional in elasticity, in the interests of that analogy.

In any realistically ice-covered ocean, the ice sheet thickness would be a very small fraction of the tidal wavelength, suggesting the use of equilibrium thin-plate theory (e.g., Landau and

Lifschitz, 1987) instead of the dynamical wave equations. That course is not followed so as to make it possible to include the interesting situation in which much thicker ice sheets are disturbed by tides, a configuration perhaps existing in the outer solar system or amongst exoplanets.

2.2 Ocean Equations and Solutions

In a non-rotating, constant density, ρ, ocean, $-d \leq z \leq 0$, and using the familiar coordinate system with u, w being *velocities* in the positive x and z directions, {ocean1}

$$-i\sigma\rho u = -\frac{\partial\left(p - g\bar{\eta}\right)}{\partial x} \tag{5a}$$ {xmomentum}

$$-i\sigma\rho w = -\frac{\partial p}{\partial z} - g\rho \tag{5b}$$ {hydrostatic}

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5c}$$ {continuity}

p' is the perturbation pressure.The system Eq. (5) is not assumed to be hydrostatic.

2.3 Non-dimensional System

Equations

With many dimensional quantities defining the system $(d, \bar{d}, \rho, \bar{\rho}, \mu, \sigma, k, g)$, it proves convenient to non-dimensionalize. A system equally useful in both the fluid and elastic media is not, however, obvious. The one chosen here is based upon the conventional time and space scales of ordinary water waves in a homogeneous fluid. Let all primed quantities be non-dimensional and,

$$T = 1/\sqrt{gk}, L = 1/k, U = L/T = \sqrt{\frac{g}{k}}, P = \rho g/k$$

$$\bar{U} = L, \bar{P} = \bar{\rho} g/k, d = d'/k, \bar{d} = \bar{d}'/k, \sigma = \sigma'\sqrt{gk}$$

$$(u, w) = U\left(u', w'\right), P = P'p', (\bar{u}, \bar{w}) = \bar{U}\left(\bar{u}, \bar{w}\right), p' = \bar{P}\bar{p}'$$

and the elastic equations become, {iceeqsnondim}

$$\frac{\partial^2 \bar{u}'}{\partial t'^2} = -\frac{\rho}{\bar{\rho}}\frac{\partial \bar{p}'}{\partial x'} + \frac{1}{\gamma^2}\nabla'^2\bar{u}' + H'\frac{\partial e^{ik'x'}}{\partial x'} \tag{6a}$$

$$\frac{\partial^2 \bar{w}'}{\partial t'^2} = -\frac{\rho}{\bar{\rho}}\frac{\partial \bar{p}'}{\partial z'} + \frac{1}{\gamma^2}\nabla'^2\bar{u}' - 1 \tag{6b}$$

$$\frac{\partial \bar{u}'}{\partial x} + \frac{\partial \bar{w}'}{\partial z} = 0 \tag{6c}$$

with $\gamma^2 = \bar{\rho} g/\mu k = \left(\sqrt{g/k}/c_s\right)^2$, $c_s = \sqrt{\mu/\rho}, H' = Hk, k' = 1$. c_s is the shear wave-speed in the ice.

The corresponding non-dimensional fluid equations are then, {nondimeqswate

$$\frac{\partial u'}{\partial t'} = -\frac{\partial p'}{\partial x'} + H' \frac{\partial e^{ik'x'}}{\partial x'} \tag{7a}$$

$$\frac{\partial w'}{\partial t'} = -\frac{\partial p'}{\partial z'} - 1 \tag{7b}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z'} = 0 \tag{7c}$$

Although $k' = 1$, it is generally displayed below as a useful marker.

Boundary Conditions

Several (dimensional) boundary conditions must be considered. At the rigid sea floor, $z = -d$,

$$w\left(-d\right) = 0.$$

At the ice-water interface, $z = \eta$, continuity of vertical displacement requires,

$$\bar{w}\left(0\right) = w\left(0\right) / \left(-i\sigma\right) = \eta$$

and which has been linearized about $z = 0$ as done in conventional wave theories. The conventional water-wave dynamic boundary condition becomes one of continuity of normal stress,

$$-\bar{p}\left(0\right) + 2\mu \frac{\partial \bar{w}\left(0\right)}{\partial z} + g\bar{\rho}\bar{w}\left(0\right) = -p\left(0\right)$$

At the top of the ice-layer, at $z = \bar{d}$, the tangential stress must vanish,

$$\frac{\partial \bar{u}\left(\bar{d}\right)}{\partial z} + \frac{\partial \bar{w}\left(\bar{d}\right)}{\partial x} = 0$$

again linearized about $z = \bar{d}$. Also, the normal stress must vanish,

$$-\bar{p}\left(\bar{d}\right) + 2\mu \frac{\partial \bar{w}\left(\bar{d}\right)}{\partial z} + g\bar{\rho}\bar{w}\left(\bar{d}\right) = 0.$$

Non-dimensionalizing as in the equations of motion, {bcs_ice_ocean

$$0 = w'\left(-d'\right), \tag{8a}$$ {nondimbcs}

$$\bar{w}'\left(0\right) = \eta' / \left(-i\sigma'\right), \tag{8b}$$

$$-\bar{p}'\left(0\right) + \frac{2}{\gamma^2} \frac{\partial \bar{w}'\left(0\right)}{\partial z'} + \bar{w}'\left(0\right) = -p', \tag{8c}$$

$$\frac{\partial \bar{u}'\left(\bar{d}\right)}{\partial z'} + \frac{\partial \bar{w}'\left(\bar{d}\right)}{\partial x'} = 0, \tag{8d}$$

$$-\bar{p}'\left(\bar{d}\right) + \frac{2}{\gamma^2} \frac{\partial \bar{w}'\left(\bar{d}\right)}{\partial z'} + w'\left(\bar{d}\right) = 0 \tag{8e}$$

From here on, the primes will be dropped, and unless otherwise specifically stated, all variables are non-dimensional (but figures are mainly presented in dimensional form).

3 Ocean Alone

For reference purposes, it proves helpful to first solve the tide problem in an ocean without an overlying ice sheet, and in an ice sheet without an underlying ocean, stuck fast to the half-space. Starting with the ocean alone, the problem is one in standard, forced, wave theory (Lamb, 1932; Kundu and Cohen, 2008), but is written out here to emphasize the parallel development in the ice.

Assuming conventional irrotational motion (tidal forcing has no curl), write $(u, w) = \nabla\varphi$, then

$$\varphi = \frac{\rho}{i\sigma}\left(p' - g\eta_{eq}\right) - \frac{g\rho z}{i\sigma}$$

and thus,

$$\nabla^2 p = g\nabla^2\eta_{eq} = -g\rho k^2 H e^{ikx}$$

with solution,

$$p = -g\rho z + \rho g H e^{ikx} + \rho\left(Ee^{kz} + Fe^{-kz}\right)e^{ikx} \tag{9}$$ {pocean1}

with,

$$\varphi = \frac{\rho}{i\sigma}\left(Ee^{kz} + Fe^{-kz}\right)e^{ikx}. \tag{10}$$ {phiocean1}

The boundary condition $w\,(z = -d) = \partial\varphi\,(-d)\,/\partial z = 0$ requires, $F = Ee^{-2kd}$ and then,

$$p\,(x, z) = -g\rho z + g\rho H e^{ikx} + \rho E\left(\cosh k\,(z + d)\right)e^{ikx} \tag{11}$$ {p1}

$$\varphi\,(x, z) = \frac{\rho E}{i\sigma}\cosh\left(k\,(z + d)\right)e^{ikx} \tag{12}$$ {phi1}

absorbing a constant factor into E.

Without overlying ice, the linearized free surface boundary conditions are,

$$-i\sigma\eta\,(x) = w\,(0) = \frac{kE}{i\sigma}\left(\sinh kd\right),$$

and Eq. (8ac) becomes $p = 0$ or,

$$-\eta + H + i\sigma\cosh kd = 0, \tag{13}$$ {surfbc1}

which is,

$$[-\sigma\cosh kd + k\sinh kd]E = -i\sigma H \tag{14}$$ {surfbc2}

If $H = 0$, the free solution produces the usual dispersion relationship for free surface waves, here

$$\sigma^2 = k\sinh kd, \quad k = 1,$$

and which would lead to resonance in Eq. (14).

Setting dimensional $k = 2\pi/6.3 \times 10^6$ (wavelength equal to the radius of the earth), and dimensional frequency as $\sigma = 2\pi/12.42\mathrm{h}$ (the modern M_2 tide), Fig. 2 shows the ordinary tidal response, as a function of dimensional depth d, direct at low frequencies, inverted at high frequencies (small d) and a transition across resonance. The limit $kd \to 0$, shallow water, is readily used if desired.

4 Ice Alone

Consider an elastic ice sheet subject to tidal forcing overlying a rigid half-space. Two reasons motivate this approach: (1) To understand the direct response of the ice to tidal forcing and (2) To understand the sensitivity of that response to boundary conditions at the ice bed—exploring the hypothesis that measurements of tidal response in the Antarctic ice sheet might shed some light on the conditions at the generally inaccesible base of the ice sheet.

Absent any $y-$dependence—as is being assumed here—the displacements in the ice can be written generally as,

$$\bar{u} = \frac{\partial \bar{\varphi}}{\partial x} + \frac{\partial \bar{\psi}}{\partial z}, \ \bar{w} = \frac{\partial \bar{\varphi}}{\partial z} - \frac{\partial \bar{\psi}}{\partial x}$$

that is as the gradient of a potential and the curl of a stream function and whose solutions are coupled through the boundary conditions. By Eq. (6c),

$$\nabla^2 \varphi = 0. \tag{15}$$ {laplace2}

(Eq. (15) is the seismological $P-$wave equation in the limit of the $P-$wave speed, $\alpha \to \infty$.)

Assume all variables are now proportional to $\exp\left(-i\sigma t\right)$. Substituting $\bar{\psi}$ in the two momentum equations, dropping the forcing term, and cross-differentiating to eliminate the pressure produces,

$$\nabla^2 \left(\nabla^2 \bar{\psi} + \sigma^2 \gamma^2 \bar{\psi}\right) = 0,$$

or integrating

$$\left(\nabla^2 \bar{\psi} + \sigma^2 \gamma^2 \bar{\psi}\right) = M\left(x, z\right), \tag{16}$$ {psi1}

where M is an harmonic function that will be set to zero. The solution then to Eq. (16) is

$$\bar{\psi}\left(x, z\right) = e^{ikx}\left(Ae^{imz} + Be^{-imz}\right), \ m = \sqrt{\sigma^2\gamma^2 - 1}$$

and the corresponding velocities are

$$\bar{u}^{\psi}\left(x, z\right) = ime^{ikx}\left(Ae^{imz} - Be^{-imz}\right), \ \bar{w}^{\psi}\left(x, z\right) = -ie^{ikx}\left(Ae^{imz} + Be^{-imz}\right).$$

Substituting back into the homogeneous momentum equations produces $\bar{p}^{\psi} = 0$, p^{ψ} being the pressure associated with the stream function.

Let the solution to Eq. (15) be

$$\bar{\varphi}(x, z) = e^{ikx}\left(Ce^{z} + De^{-z}\right)$$

Substituting φ into the non-dimensional momentum equations produces,

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\frac{\partial \bar{\varphi}}{\partial x} &= -\frac{\rho}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + He^{ikx}, \\ \frac{\partial^2}{\partial t^2}\frac{\partial \bar{\varphi}}{\partial z} &= -\frac{\rho}{\bar{\rho}}\frac{\partial \bar{p}}{\partial z} - 1 \end{aligned}$$

which leads to

$$\begin{aligned} \frac{\rho}{\bar{\rho}}\bar{p}(x, z) &= -\frac{\partial^2 \bar{\varphi}}{\partial t^2} + He^{ikx} - z \qquad (17) \\ &= \sigma^2 \bar{\varphi} + He^{ikx} - z \\ &= \sigma^2 \left(Ce^{kz} + De^{-kz}\right) e^{ikx} + He^{ikx} - z \end{aligned}$$

{pice}

The boundary conditions at $z = 0$ are now $\bar{u}(0) = 0, \bar{w}(0) = 0$, together with those at $z = \bar{d}$ are,

{iceallbc}

$$m(A - B) + k(C + D) = 0 \qquad (18a)$$

{icea}

$$-ik(A + B) + k(C - D) = 0 \qquad (18b)$$

{iceb}

$$\left(-m^2 + k^2\right)\left(Ae^{im\bar{d}} + Be^{-im\bar{d}}\right) + 2ik\left(Ce^{k\bar{d}} - De^{-k\bar{d}}\right) = 0 \qquad (18c)$$

{icec}

$$\begin{aligned} &A\left(\frac{2km}{\gamma^2} - 2ik\right)e^{im\bar{d}} + B\left(-\frac{2km}{\gamma^2} - 2ik\right)e^{-im\bar{d}} + C\left(-\sigma^2 + \frac{2k^2}{\gamma^2} + 2k\right)e^{k\bar{d}} + \\ &\qquad D\left(-\sigma^2 + \frac{2k^2}{\gamma^2} + 2k\right)e^{-k\bar{d}} = H \qquad (18d) \end{aligned}$$

{icee}

Taking μ =2.3×10^9 (Squire et al., 1995), $\gamma \approx 2.1$. The dimensional values of $\bar{\eta} = \bar{w}\left(\bar{d}\right)$ are shown in Fig. 3. For thin ice sheets, little or no tidal response occurs. As the ice thickens, the vertical displacement grows, as it does for high frequencies as the forcing phase speeds approach that for free shear waves in the ice. In the high frequency limit, a free solution which is a Rayleigh wave at the upper boundary of the ice sheet can produce a large response. Higher modes also are possible, although no real Earth tidal forcing exists with such phase speeds.

Much interest exists in the determination of the boundary conditions at the base of the ice sheet—a generally inaccessible place. The surface response of a continental ice sheet does depend upon those lower boundary conditions. For realistic Earth ice sheets, the forced response is sufficiently slight, whatever the basal boundary conditions, that the ability to measure the motions is somewhat doubtful, and thus the analysis is here placed in an Appendix.

5 Coupled Ice and Ocean

The non-dimensional system of boundary conditions can be written, {iceandoceanal

$$-ikA - ikB + kC + kD + \frac{k}{i\sigma}\sinh\left(kd\right)E = 0, \tag{19a}$$

$$\left(\frac{2}{\gamma^2}km - 2ik\right)A + \left(-\frac{2}{\gamma^2}km - 2ik\right)B + \left(-\sigma^2 + \frac{2}{\gamma^2}k^2 + 2k\right)C + \left(-\sigma^2 + \frac{2}{\gamma^2}k^2 + 2k\right)D +$$
$$\left(i\sigma\cosh kd + \frac{k}{i\sigma}\sinh kd\right)E = \left(\frac{\bar{\rho}}{\rho} - 1\right)H, \tag{19b}$$

$$\left(-m^2 + k^2\right)A + \left(-m^2 + k^2\right)B + 2ik^2C - 2ik^2D = 0, \tag{19c}$$

$$\left(-m^2 + k^2\right)\left(Ae^{im\bar{d}} + Be^{-im\bar{d}}\right) + 2ik\left(Ce^{k\bar{d}} - De^{-k\bar{d}}\right) = 0, \tag{19d}$$

$$A\left(\frac{2km}{\gamma^2} - 2ik\right)e^{im\bar{d}} + B\left(-\frac{2km}{\gamma^2} - 2ik\right)e^{-im\bar{d}} + C\left(-\sigma^2 + \frac{2k^2}{\gamma^2} + 2k\right)e^{k\bar{d}} +$$
$$D\left(-\sigma^2 + \frac{2k^2}{\gamma^2} + 2k\right)e^{-k\bar{d}} = H \tag{19e}$$

with the last two equations unchanged from those for ice-alone. Setting $\bar{\rho}/\rho = 1$ is a useful approximation in 19b. Fig. 4 shows the contours, as a function of dimensionless σ and $\bar{d}$ of η and $\bar{\eta}$ and Fig. 5 the corresponding lateral displacements.

6 Snowball Earth-Ocean

The question raised here is the nature and possible influence of tides in a snowball-Earth-like environment. In the modern ocean, tides are believed to provide a significant fraction of the energy required to sustain the observed three-dimensional circulation (Munk and Wunsch, 1998), roughly about 50%, with much of the energy used to provide the vertical mixing. Almost all of the rest comes from the wind-field—assumed absent in an ice-covered world—although this inference remains somewhat insecure owing to the complexity of the response to buoyancy forcing at top and bottom.

6.1 More Realism

Tides of a realistic ocean can be considerably more complex, involving rotation, interaction with boundaries, topography, and stratification. Some properties even of the non-rotating canal theory remain robust in the presence of all of these complications as the barotropic solutions (no stratification) remain governed by gravity-wave physics even where rotation is important. Resonances still appear, although they can be generated by the presence of sidewalls and not just from travelling-wave version seen here. Particle velocities are strongly influenced by rotation, as would the boundary-layer between the ocean and the base of the ice sheet. In a non-rotating ocean, an important mechanical boundary layer scale would be $(A/\sigma)^{1/2}$, becoming $\left(A/\sqrt{\sigma^2-f^2}\right)^{1/2}$ where $f=2\Omega\sin\phi$, Ω being the Earth's rotation rate, and ϕ the latitude and A is a hypothetical eddy-viscosity. At latitudes where $\sigma\approx f$ (the "inertial latitude"), the boundary layer physics are distinct. For semi-diurnal tide constituents, that occurs only poleward of about 70° latitude, but for diurnals it is at about 30°.

Numerous studies do exist of the boundary layer flows under ice in the Arctic (e.g., McPhee, 2002; Cole et al., 2014) where rotation tends to be important or dominant. For obvious reasons, no observations exist of low-latitude sea ice-boundary layer interactions. Under-ice topography can be very rough, and how to model the fluid interactions at low latitudes is not so clear. A reasonable inference is that dissipation at the sea ice-water boundary would be at least as important as that over abyssal planes today, and possibly considerably greater.

Of principle concern in discussing a snowball Earth is the topographic change: modern day tides have a substantial fraction of their dissipation occurring in the shallow regions of the continental margins (Egbert and Ray, 2001). Tidal response in shallow water, $d' << d$, is largely a "co-oscillation" forced by the incoming tide from deeper water, rather than being a direct response to the local forcing. In an ice-covered ocean with shallow margins, the deep water tide would tend to undergo reflection as the ice-lid becomes ever-more effective as $d'/\bar{d}$ vanishes, and it is a reasonable surmise that continental margin dissipation would be greatly reduced relative to today's values (See Fig. 7.)

The second major tidal dissipation mechanism in the modern ocean is through baroclinic conversion from the stratification and the presence of topography (e.g., Egbert and Ray, 2000). If the snowball ocean is nearly unstratified as in the A2014 results, baroclinic conversion would also be much reduced. Thus both major dissipation mechanisms become weaker, and the perhaps paradoxical inference is that tides of an ocean with an ice-lid are likely to be considerably stronger than they are today. An important caveat is that proximity to resonance can be a sensitive function of the continental configuration and which will have changed greatly through millions of years.

6.2 Influence on the Circulation

Consider the energetics of a snowball Earth ocean. The A2014 thermal forcing of 0.1W/m^2 corresponds to a net power input of 3.6×10^{13}W, (36 Terrawatts, TW), an impressive amount of energy compared to estimates of the energy required to maintain the modern ocean circulation of roughly 2TW. On the other hand, the A2014 solutions depict a circulation with a thermal range of about 0.4°C, and so the Carnot efficiency would be about 0.4/273= 0.0015, reducing the useable power to about 50GW. This value is probably an upper bound on the efficiency (e.g., Peixoto and Oort, 1992). Is it possible that the tides of such an ocean would be energetically competitive? Tidal forcing, in contrast, is a direct mechanical driver of kinetic energy; whether a significant large-scale time-mean circulation is generated in practice has to be separated from the question of overall energy input and dissipation.

In the modern ocean, particularly in shallow water, significant residual circulations result from strong tidal flows (e.g. barotropic ones, Maier-Reimer, 1977; Zimmerman, 1978; and baroclinic, King et al., 2009; Xing et al., 2011; Grisouard and Bühler, 2012). Continuing down this speculative path, one might infer that shallower regions of a snowball ocean would produce significant tidally-driven circulations. These would necessarily interact with any circulation also present from convective driving. In general, the strength of rectified flows is inversely proportional to the square of the water depth (e.g., Zimmermann, 1978) and would thus depend upon just how much residual ocean water remained, as well as upon the bottom topography. Is it possible that the circulation established by a barotropic or baroclinic tidal flow could compete with that driven by the geothermal heating? Without actually answering that question, note that a 1mm/s meridional barotropic flow, extending the width of the present Pacific Ocean (10,000km) in a water depth of 2km, would produce a transport of about 20Sv, as compared to the 30Sv maximum estimated by A2014 for the geothermal response.

Oceanic general circulation models usually have covert sources of energy, hidden in the various sub-grid-scale eddy-mixing schemes. Explicit energetics would govern the breakdown of an eddy-field derived from instabilities of the larger-scale flows. Other processes, such as those generally ascribed to the breaking of internal waves, internal tides and related phenomena, would however, have power sources hidden in eddy mixing coefficients. Most of this physics requires a stratified fluid, and as the A2014 ocean is nearly unstratified, the role of tidal mixing is far from obvious.

7 Other Processes

For obvious reasons, none of the results as applied to the snowball Earth are definitive, and many unknowns and complications intervene. Some interesting physics problems arise. Among other intriguing complications not discussed here are the role of the changed Earth rotation rate and length of the month at times approaching -1GY when the day was probably about 22 hours long, and with about 13 synodic months in the year (Bills and Ray, 1999; Williams, 2000). These changes are consequences of tidal friction and the resulting braking of the Earth's spin over time.

If the modern ocean depth is reduced by half, and assuming that 600 million years ago that the salt amount in the ocean was similar to today, salinity would have roughly doubled to about 7% of the water mass. A salty fluid, heated from below can be unstable to double-diffusive processes (see e.g., Turner, 1973; Brandt and Fernando, 1995) forming a layered circulation. Whether over millions of years that possibility persists, and what would be the consequences of any annual cycling at low latitudes in the ice-cover, has not been discussed. If some stratification does persist, then baroclinic conversion from the barotropic tide can occur, a spatially dependent mixing would arise, and a whole suite of theoretical problems can be defined including the baroclinic mean flows already alluded to above. Whether any possibility exists of an observational test of such interesting configurations is unclear, and we leave the problem as one of near-total speculation.

Appendix. Basal Boundary Condition Sensitivity

The equations governing an ice sheet frozen to a rigid underlying half-space are in Eqs. (**??**). If the the no-slip boundary condition (Eq. 18a) is replaced by one of no shear stress,

$$A\left(-m^2+k'^2\right)+B\left(-m^2+k'^2\right)+C\left(2ik'^2\right)+D\left(-2ik'^2\right)=0.$$

Fig. 8 displays the dimensionalized solutions $\bar{u}\left(\bar{d}\right)$ for an M_2 tidal forcing for the two boundary conditions. No-slip lateral motion is only about 1% of that for free-slip for realistic thicknesses which in principle would permit determination of the the appropriate boundary condition for surface displace (shear) measurements. On the other hand, the lateral displacement is of the order of microns, even for free slip, and the feasibility of detecting such small values in the presence of Earth noise is obscure.

Acknowledgements

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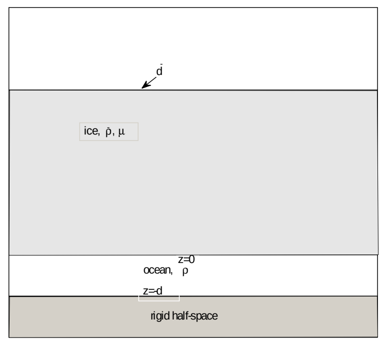

Figure 1: Defining schematic. {sketch.eps}

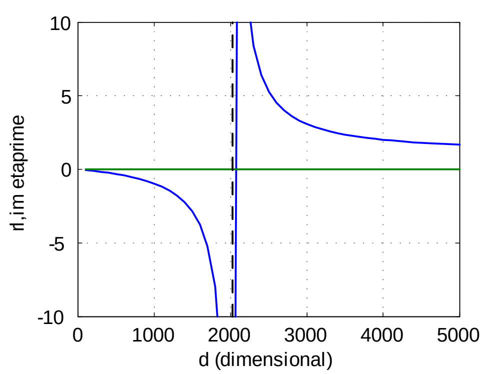

Figure 2: η (dimensional) for the ocean alone, with the resonance (vertical dashed line) at $\sigma/k = \sqrt{gd}$ apparent. Amplitude scale is truncated at the resonance. {ocean_alone_e

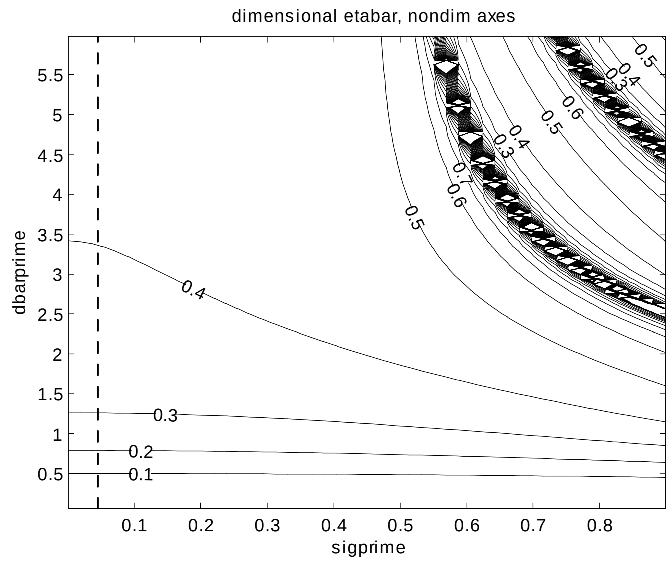

Figure 3: Contours of $\bar{w}\,(\bar{d})$ the vertical displacement at the top of the ice layer. Vertical dashed line is the M_2 tide frequency, far from the resonances visible in the upper-right corner.

{ice_along_dba

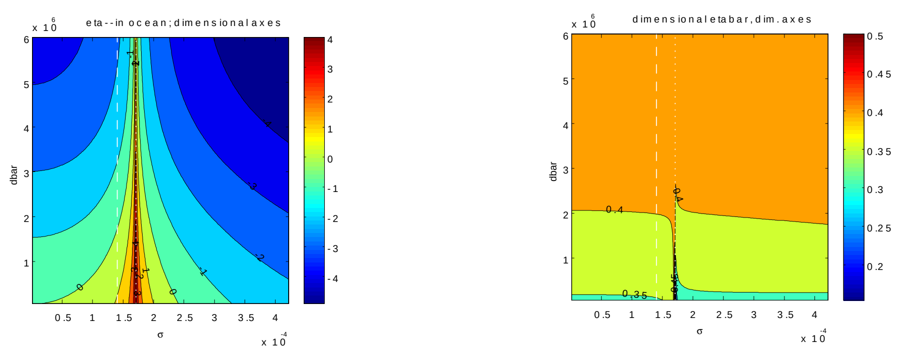

Figure 4: $\log_{10}(\eta)$, (left panel) in dimensional form and for $\bar{\eta}$ i with $d = 3000$m. White dashed line is the tidal frequency and dotted line is the resonance frequency.

{eta&etabar_mo

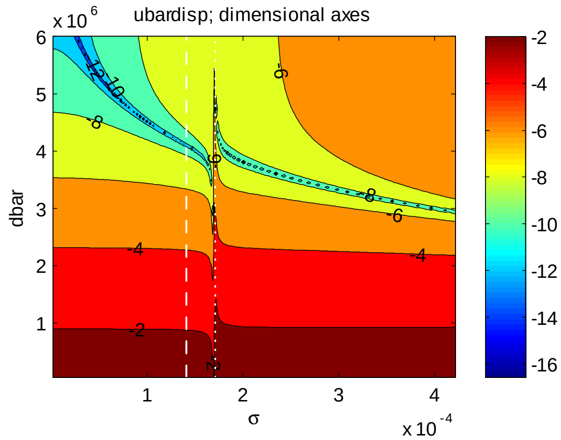

Figure 5: $\log_{19}(\bar{u}\left(z=\bar{d}\right))$ $d = 3000$m with dimensional axes.

{ubardisp_ice&

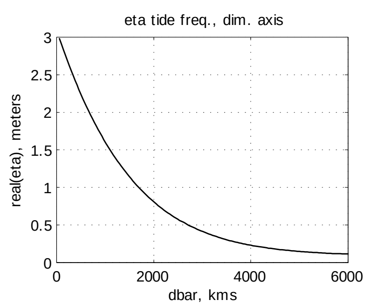

Figure 6: $\eta\left(\bar{d}\right)$ at the tidal frequency, showing the decline in the ocean tide as the ice thickness grows. A value of 3m at $\bar{d} = 0$ is the result of near-resonant amplification of the ocean equilibrium value (1m). $d = 3000$m.

{eta_withice_d

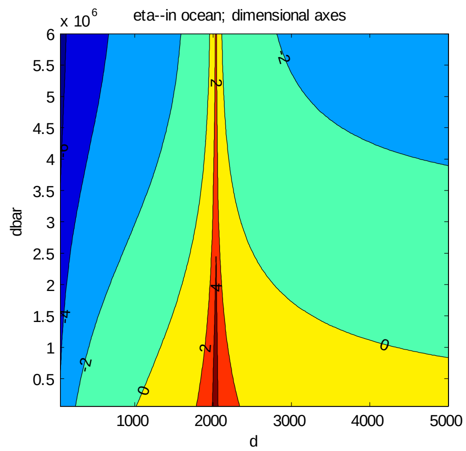

Figure 7: $\log_{10}(\eta)$ as a function of $\bar{d}, d$ for the M_2 tidal frequency. For fixed d, the response diminishes with increasing ice thickness. The resonance, dominated by the ocean depth change is apparent. For fixed $\bar{d}$, η diminishes rapidly as the ocean depth is reduced and which would lead to reflection of incoming energy. {eta_ice&ocean

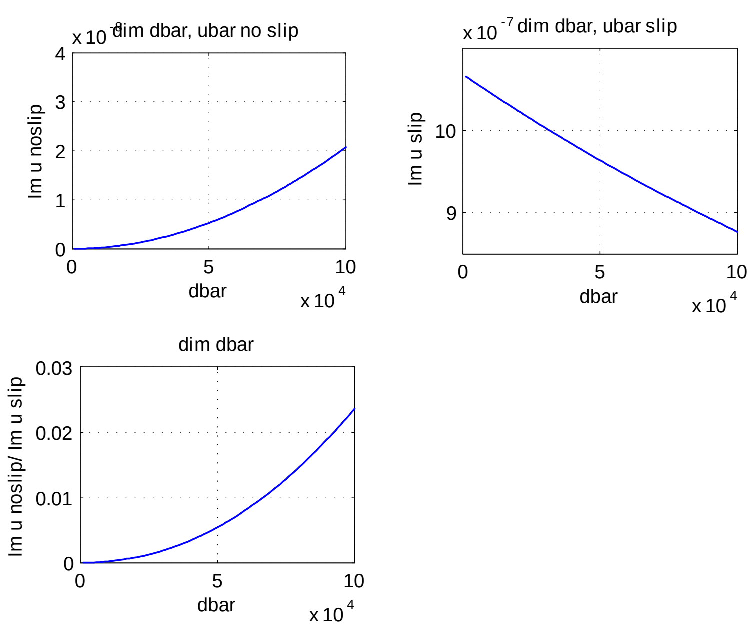

Figure 8: Dimensional $\bar{u}\left(\bar{d}\right)$ versus $\bar{d}$ for boundary conditions of no slip (upper left panel), free slip—no shear stress (upper right) and their ratio in an ice sheet in contact with a rigid half-space at $z = 0$. {slip_noslip_u