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Note on some Reynolds stress effects of internal waves on slopes

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Abstract—When two-dimensional internal waves encounter a shoaling region, the second order mass and momentum flux carried by the waves must be re-adjusted if no mixing occurs.The re-adjustment occurs by a 'set-up' and 'set-down' of the mean isopycnals generating a second order mean Eulerian flow to cancel the Stokes drift velocity. Since the effects of Coriolis forces in an infinite flat bottomed channel are analogous to buoyancy forces on a sloping bottom, rotation is easily taken into account, however the problem is degenerate to an arbitrary long-shore geostrophic flow.

1. INTRODUCTION

THE EFFECTS of second order momentum flux in surface waves are well known (LONGUET-HIGGINS and STEWART, 1964) and account for non-negligible effects in, for example, breaking waves. Similar effects should occur in internal waves, but these do not appear to have been examined in any detail. THORPE (1968), BRETHERTON (1969), and others have treated some aspects of non-linear effects on internal waves. This particular study was motivated by the observation that in the vicinity of Bermuda, there appear to be systematic deviations of the mean isopycnals from open ocean conditions as the island is approached (WUNSCH, 1971; HOGG, 1971), and it was decided to investigate the possibility that these deviations could be accounted for by 'radiation stress' effects similar to the wave set-up and set-down effects of surface waves (LONGUET-HIGGINS and STEWART, 1964). Of course, any such effects, if non-negligible, would have to compete with other processes, including the effects of the island on the mean flow past it (HOGG, 1971). HASSELMANN (1970) has treated a problem closely analogous to this one (see below).

In this note, we will confine ourselves to investigating the effects of 2-dimensional internal waves encountering a shoaling beach. The extension to 3-dimensions as well as more general conditions of stratification and topography have been treated by HOGG (1971). Getting much beyond order of magnitude estimates necessarily involves difficult questions of dissipation mechanisms for internal waves.

The basic physics is easily demonstrated. Consider an infinite channel of constant depth d, constant Brunt frequency N, and rigid top and bottom. Then the linear equation governing internal wave propagation in a Boussinesq fluid is:

$$\psi_{zz} - \frac{1}{c^2}\, \psi_{xx} = 0,\ c^2 = \frac{\omega^2}{N^2 - \omega^2}\,, \tag{1}$$

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with solutions:

$$\psi = \sin \frac{n\pi z}{d} \cos \left(\frac{n\pi}{d} cx + \omega t\right), \tag{2}$$

where ψ is the stream function such that $u = -\psi_z$, $w = \psi_x$ in the notation of WUNSCH (1969), u the x component of $\vec{u}$ and w the z component. ω is the wave frequency.

Following PHILLIPS (1966) or LONGUET-HIGGINS (1969), we can, for infinitesimal waves (defined explicitly below), find the Lagrangian velocity

$$\vec{u}_L(a, t) = \vec{u}_E(\vec{a}, t) + \left(\int_0 \vec{u}_E(\vec{a}, t')dt'\right) \cdot \nabla_a \vec{u}_E(\vec{a}, t), \tag{3}$$

for small times t, where $\vec{a}$ is the Lagrangian initial position tag, and $\vec{u}_E$ is the infinitesimal Eulerian velocity given by the above stream function. The second term on the right is the 'Stokes velocity'. With LONGUET-HIGGINS (1969), we write $\vec{u}_L = \vec{u}_E + \vec{u}_S$. If the wave amplitudes are small of $0(\varepsilon)$ where ε is an ordering parameter, the Stokes velocity is $0(\varepsilon^2)$. Hence, to evaluate the Lagrangian velocity to $0(\varepsilon^2)$, we must, for consistency, evaluate the Eulerian velocity to $0(\varepsilon^2)$ as well. If we time average (3), we have:

$$\langle \vec{u}_L(a, t) \rangle = \langle u_E(a, t) \rangle + \langle \vec{u}_S(a, t) \rangle, \tag{4}$$

where the angle brackets denote a time average. It is easy to show that $\langle \vec{u}_E \rangle = 0$ to $0(\varepsilon^2)$ for these internal waves in a channel, hence:

$$\langle u_L \rangle = \langle \vec{u}_S \rangle.$$

Doing the indicated integrations and time averages in (4), we find:

$$\langle \vec{u}_L \rangle = -\left(\frac{n\pi c}{d}\right)^3 \frac{1}{2\omega c^3} \cos \frac{2n\pi z}{d}, \langle w_L \rangle = 0. \tag{5}$$

Note that the average over depth:

$$\int_{-d}^{0} \vec{u}_L dz = 0,$$

hence, channeled internal waves carry no *net* momentum to this approximation. At any given level (with the exception of the nodes) however, they do carry finite momentum and mass. In Fig. 1 is shown the velocity distribution of the mean drift for the fundamental mode. This Stokes drift is, of course, generated for the reason that there is a Stokes drift in a surface wave; the particle orbits are not of zero size. An individual particle does not quite close its orbit owing to the spatial inhomogeneities in velocity; the resulting drift carries the net wave momentum. Note however, that a current meter, an essentially Eulerian device, would not measure a mean drift, but only the purely harmonic Eulerian component (LONGUET-HIGGINS, 1969).

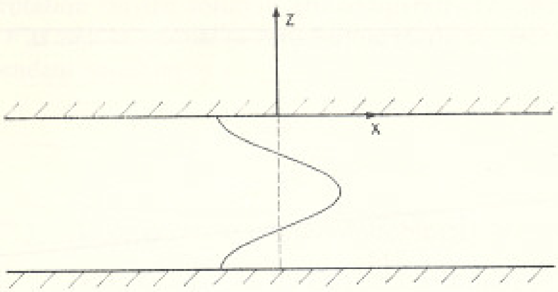

Fig. 1. Mean Langrangian drift velocity for the fundamental channel mode $n = 1$. Units are arbitrary. The wave is propagating from right to left.

Consider now what happens if a 'beach' is placed at one end of the channel. Every isopycnal intersects the slope and hence the particles drifting along the isopycnals also would eventually intersect the slope. If there is no mixing, the particles cannot leave their original isopycnals and hence, the Stokes drift cannot occur anywhere in the channel if there is a beach at one end. The fluid must re-adjust itself throughout the length of the channel in such a way as to make the average Lagrangian velocity zero everywhere. The way in which this adjustment is brought about is by generation of a non-zero mean Eulerian velocity that is equal and opposite to the Stokes velocity; this Eulerian velocity is driven by a pressure gradient formed by a re-adjustment of the mean buoyancy field by the gradients of the wave Reynolds stresses. The purpose of the remaining sections of this note is to work out the details of this process and to estimate its possible importance in the ocean.

Note that in an internal wave channel with a flat bottom, there is a similar process if Coriolis forces are taken into account. In a rotating frame, there is no force in the y-direction to balance Coriolis forces acting on the Stokes drift velocity (5). The only method by which the fluid can maintain a y-force balance is through the Reynolds stresses. These stresses generate an Eulerian mean velocity equal and opposite to (5). This is the mechanism analyzed by HASSELMANN (1970) for the transients created by a wave train. In what follows, the equations will be set up to make the analogies between the two situations clear.

2. WAVES ON A SLOPE

WUNSCH (1969) showed that the linearized exact wave solutions on a uniformly sloping beach were asymptotically equal to the channel modes (2) if the beach slope were sufficiently small and the wave field evaluated sufficiently far from the corner formed by the beach intersecting the surface. Thus, by using these solutions, we can explore the effects of a beach at the far end of a long channel in the absence of mixing.

The coordinate system is shown in Fig. 2. The velocity field notation is conventional, with v^* along the beach. σ^* is the buoyancy such that

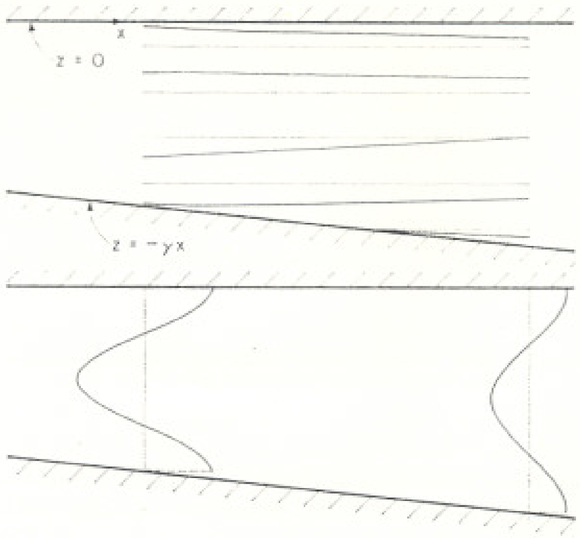

Fig. 2. Coordinate system for the wedge geometry. The mean displacement of the isopycnals for mode 1 (upper) is shown as well as the mean Eulerian horizontal velocity for the case $c = 1$, $\gamma = 1{\cdot}1$ Units are arbitrary. The wave is propagating from right to left and $F = 0$. Light lines are horizontal.

$$\sigma^* = g\,\frac{(\rho^* - \rho_o^*)}{\rho_o^*},$$

where * denotes a dimensional quantity, and ρ_o^* is a constant reference density. We non-dimensionalize as follows:

$$\psi^* = a\,\psi,\ \sigma_o^* = \frac{a\,N}{L}\,\sigma,\ v_o^* = \frac{a}{L}\,v,$$

with the notation $\varepsilon = a/NL^2$, $F = f/N$ where N is the Brunt frequency and f, the Coriolis parameter. We have the following non-dimensional equations in the Boussinesq approximation with no y-dependence and where we are linearizing about a basic stratification $\partial\sigma_o/\partial z = -\tilde{N}^2$

$$-\partial/\partial t(\nabla^2\psi) + \varepsilon\left(\frac{\partial\psi}{\partial z}\,\partial/\partial x - \frac{\partial\psi}{\partial x}\,\partial/\partial z\right)\nabla^2\psi - \mathrm{F}\,\frac{\partial v}{\partial x} = \frac{\partial\sigma}{\partial x} \tag{6}$$

$$\frac{\partial v}{\partial t} + \varepsilon\left(\frac{\partial\psi}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial v}{\partial x}\right) - F\,\frac{\partial\psi}{\partial z} = 0, \tag{7}$$

$$\frac{\partial\sigma}{\partial t} + \varepsilon\left(\frac{\partial\psi}{\partial x}\frac{\partial\sigma}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\sigma}{\partial x}\right) - \tilde{N}^2\,\frac{\partial\psi}{\partial x} = 0. \tag{8}$$

The length scale L is to be thought of as a local depth in the wedge-shaped region formed by the beach. ε is thus a measure of the wave displacement relative to the local depth and will be a perturbation parameter.

The effects of rotation on the solution are comparatively uninteresting, so for clarity, we will put F as well as v equal to zero for the moment, restoring them below. Expanding the dependent variables in ε,

$$\psi = \psi^{(0)} + \varepsilon\,\psi^{(1)} + \dots$$

$$\sigma = \sigma^{(0)} + \varepsilon\,\sigma^{(1)} + \dots,$$

and assuming $\langle \psi^{(0)} \rangle = 0$, we generate a series of problems. To Order 1:

$$\frac{\partial}{\partial t}\,\sigma^{(0)} = \tilde{N}^2\,\frac{\partial \psi^{(0)}}{\partial x}, \tag{9}$$

$$-\frac{\partial}{\partial t}(\nabla^2 \psi^{(0)}) = \frac{\partial \sigma^{(0)}}{\partial x}. \tag{10}$$

To Order ε:

$$\frac{\partial \sigma^{(1)}}{\partial t} + \left(\frac{\partial \psi^{(0)}}{\partial x}\frac{\partial \sigma^{(0)}}{\partial z} - \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial \sigma^{(0)}}{\partial x}\right) = \tilde{N}^2\,\frac{\partial \psi^{(0)}}{\partial x}, \tag{11}$$

$$-\frac{\partial}{\partial t}(\nabla^2 \psi^{(1)}) + \left(\frac{\partial \psi^{(0)}}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \psi^{(0)}}{\partial x}\,\partial/\partial z\right)\nabla^2 \psi^{(0)} = \frac{\partial \sigma^{(1)}}{\partial x}. \tag{12}$$

Combining equations (9) and (10), we have:

$$\frac{\partial^2 \psi^{(0)}}{\partial z^2} - \left(\frac{\tilde{N}^2 - \omega^2}{\omega^2}\right)\frac{\partial^2 \psi^{(0)}}{\partial x^2} = 0, \tag{13}$$

subject to boundary conditions $\psi = 0$ on $z = 0$, and on $z = -\gamma x$, $x > 0$ whose solution in the present non-dimensional form was given by Wunsch (1969) and is:

$$\psi^{(0)} = \cos\left[q \ln(cx - z) + \omega t\right] - \cos\left[q \ln(cx + z) + \omega t\right] \tag{14}$$

$$q = \frac{2n\pi}{\ln\left(\dfrac{c+\gamma}{c-\gamma}\right)},\ c^2 = \frac{\omega^2}{\tilde{N}^2 - \omega^2},\ n = \text{integer},$$

and for which $\gamma < c$ where γ is the beach slope and c, the internal wave characteristic slope. We also find:

$$\sigma^{(0)} = \frac{\tilde{N}^2}{\omega}\left\{\frac{qc}{cx - z}\cos\left[q \ln(cx - z) + \omega t\right] - \frac{qc}{cx + z}\cos\left[q \ln(cx + z) + \omega t\right]\right\}. \tag{15}$$

At the next order, only time averages are needed. Averaging (11), (12), we have:

$$\frac{\partial}{\partial x}\langle \psi^{(1)} \rangle = \left\langle \frac{1}{\tilde{N}^2}\,\frac{\partial \psi^{(0)}}{\partial x}\frac{\partial \sigma^{(0)}}{\partial z} - \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial \sigma^{(0)}}{\partial x} \right\rangle \tag{16}$$

$$\frac{\partial}{\partial x}\langle \sigma^{(1)} \rangle = \left\langle \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial}{\partial x}\nabla^2 \psi^{(0)} - \frac{\partial \psi^{(0)}}{\partial x}\frac{\partial}{\partial z}\nabla^2 \psi^{(0)} \right\rangle. \tag{17}$$

We need to evaluate quantities such as:

$$\langle \sin(q \ln(cx - z) + \omega t) \cos(q \ln(cx - z) + \omega t) \rangle = \frac{1}{2} \sin\left((q \ln\left(\frac{cx - z}{cx + z}\right)\right) \quad (18)$$

and

$$\langle \sin(q \ln(cx - z) + \omega t) \sin(q \ln(cx + z) + \omega t) \rangle = \frac{1}{2} \cos\left(q \ln\left(\frac{cx - z}{cx + z}\right)\right). \quad (19)$$

From (16), we find:

$$\frac{\partial}{\partial x} \langle \psi^{(1)} \rangle = \frac{2q^2c^2}{\omega(c^2x^2 - z^2)^2} \left\{ qz\cos\left[q\ln\left(\frac{cx-z}{cx+z}\right)\right] - cx \sin\left[q \ln\left(\frac{cx-z}{cx+z}\right)\right]\right\} \quad (20)$$

and from (17):

$$\langle \sigma^{(1)} \rangle = \frac{q^2(c^2+1)}{(c^2x^2 - z^2)^2} \left\{ z\cos\left[q\ln\left(\frac{cx-z}{cx+z}\right)\right] + qcx\sin\left[q\ln\left(\frac{cx-z}{cx+z}\right)\right]\right\}. \quad (21)$$

Note that (20) is the second order *Eulerian* mean vertical velocity—a quantity that vanishes over a flat bottom. We will temporarily postpone the determination of the second order Eulerian mean horizontal velocity, the calculation of which appears to involve integration of (20).

The appearance of the mean change in buoyancy at $0(\varepsilon)$ is a direct consequence of the need for a pressure gradient in the fluid to balance the Lagrangian drift velocity. Note that the effect diminishes as $1/x^3$ away from the corner. In Fig. 2 is shown the mean isopycnal displacement as generated by the first mode ($n = 1$). To retain the validity of the scaling, the expression (21) must not exceed $0(1)$, or roughly, $q/c < x$.

If the rationalization given in the introduction is correct, the mean Stokes velocity computed from (4) should be equal and opposite to (20). Doing the required algebra, we do find indeed that $\langle w_s \rangle = -\frac{\partial}{\partial x} \langle \psi^{(1)} \rangle$. In addition, we can find u_s:

$$\langle u_s \rangle = \frac{2q^2c}{\omega(c^2x^2 - z^2)^2} \left\{ -qcx\cos\left[q\ln\left(\frac{cx-z}{cx+z}\right)\right] + z\sin\left[q\ln\left(\frac{cx-z}{cx+z}\right)\right]\right\}, \quad (22)$$

and by somewhat laborious algebra, it may be confirmed that in fact this is the required integral and re-derivation in z to obtain $\partial/\partial z \langle \psi^{(1)} \rangle$. Thus, we have confirmed that the mean Lagrangian velocities are zero. Note that the second order velocity field satisfies both the upper and lower boundary conditions.

Assume now that F is the order of, but less than, one, to preserve an internal wave range. The sequence of problems is now:

$$0(1): \quad \frac{\partial \sigma^{(0)}}{\partial t} = \frac{\bar{N}^2 \partial \psi^{(0)}}{\partial x} \quad (23)$$

$$\frac{\partial v^{(0)}}{\partial t} = F \frac{\partial \psi^{(0)}}{\partial z} \quad (24)$$

$$-\frac{\partial}{\partial t}(\nabla^2\psi^{(0)}) - F\frac{\partial v^{(0)}}{\partial z} = \frac{\partial \sigma^{(0)}}{\partial x}, \tag{25}$$

$$0(\varepsilon): \quad \frac{\partial \sigma^{(1)}}{\partial t} + \left(\frac{\partial \psi^{(0)}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial}{\partial x}\right)\sigma^{(0)} = \tilde{N}^2\frac{\partial \psi^{(1)}}{\partial x} \tag{26}$$

$$\frac{\partial v^{(1)}}{\partial t} + \left(\frac{\partial \psi^{(0)}}{\partial x}\frac{\partial}{\partial z} - \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial}{\partial x}\right)v^{(0)} = F\frac{\partial \psi^{(1)}}{\partial z}, \tag{27}$$

$$-\frac{\partial}{\partial t}(\nabla^2\psi^{(1)}) + \left(\frac{\partial \psi^{(0)}}{\partial z}\frac{\partial}{\partial x} - \frac{\partial \psi^{(0)}}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\psi^{(0)} - F\frac{\partial v^{(1)}}{\partial z} = \frac{\partial \sigma^{(1)}}{\partial x}. \tag{28}$$

Note that if we formally put σ and $N=0$ in these equations, they are identical to those of (9) to (12) when the substitutions listed by VERONIS (1967) are made, reflecting the symmetry between rotating and stratified systems.

The order one problem leads to the same solution as (14) and (15), except now $c^2=(\omega^2-F^2)/(\tilde{N}^2-\omega^2)$ and we have:

$$v^{(0)} = \frac{-qF}{(cx-z)\omega}\cos[q\ln(cx-z)+\omega t] - \frac{qF}{(cx+z)\omega}\cos[q\ln(cx+z)+\omega t].$$

The time averages of (26) to (28) lead to:

$$\tilde{N}^2\left\langle \frac{\partial \psi^{(1)}}{\partial x}\right\rangle = \left\langle \frac{\partial \psi^{(0)}}{\partial x}\frac{\partial \sigma^{(0)}}{\partial z} - \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial \sigma^{(0)}}{\partial x}\right\rangle, \tag{29}$$

$$F\left\langle \frac{\partial \psi^{(1)}}{\partial z}\right\rangle = \left\langle \frac{\partial \psi^{(0)}}{\partial x}\frac{\partial v^{(0)}}{\partial z} - \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial v^{(0)}}{\partial x}\right\rangle, \tag{30}$$

$$\left\langle \frac{\partial \psi^{(0)}}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi^{(0} - \frac{\partial}{\partial x}\psi^{(0)} \;\partial/\partial z\,\nabla^2\psi^{(0)}\right\rangle = F\left\langle \frac{\partial v^{(1)}}{\partial z}\right\rangle + \left\langle \frac{\partial \sigma^{(1)}}{\partial x}\right\rangle. \tag{31}$$

It may be easily confirmed that (29) and (30) are consistent and lead back to (20) and (22) as before, with the new definition of c. Now both the Coriolis forces and buoyancy forces outlined in the introduction act to bring the mean Lagrangian velocity down to zero. Equation (31) expresses the combined change in buoyancy due to the Reynolds stresses, plus a buoyancy change due to the thermal wind. Note that to this order, we can add an arbitrary steady v and σ, such that

$$-F\frac{\partial\langle v\rangle}{\partial z} = \frac{\partial}{\partial x}\langle\sigma\rangle$$

without changing the Reynolds stress anywhere. Thus, the problem is degenerate; one cannot tell how much of the isopycnal slope is due to the Reynolds stresses, and how much to a long-shore, geostrophic flow.

3. MAGNITUDE OF EFFECTS

The dimensional magnitude of (21) is $N^2L\varepsilon^2$. Taking an oceanic value of $N \sim 10^{-3}$, a depth of water of $L = 1\text{km} = 10^5$ cm, and choosing $\varepsilon \sim 10^{-1}$, equivalent to an internal wave amplitude of 100 m, then with the relation:

$$\Delta z^* = \frac{\Delta\sigma^*}{N^2},$$

we find a displacement Δz of the isopycnals of $0(10^3)$ cm, an observable amount in principle. By letting $\varepsilon \to 1$ of course, the effects become extremely large, but the perturbation method fails then. This result does indicate, though, the potentially very large effects of shoaling internal waves.

The dimensional Eulerian mean velocities are of order $(a/L)\ \varepsilon = NL\ \varepsilon^2$, or about 1 cm/sec a small but measureable amount in the absence of other mean currents. Again, by pushing ε toward 1, the effects are potentially very large.

The most serious objection to the theory outlined here is the complete neglect of dissipation. The linearized solutions (14) formally represent propagation of the waves into the corner, where the energy density becomes infinite. On the other hand, the criterion of linearity, that ε be small locally, is violated when $x < q/c$. Observations in the laboratory by CACCHIONE (1970) and inference at Bermuda by WUNSCH (1971), indicate that the solutions break down by releasing energy to mixing either in the boundary layer at the bottom, or overturning of the waves themselves in a breaking mechanism. Once this breakdown occurs of course, the constraint generating the 'set' effect is also broken, at least partially. If the mixing occurs high up on the slope, as it seems to in the laboratory, then the set-up effects occur below the isopycnal that demarcates the lower edge of the mixing zone. Above this isopycnal there would be a Lagrange drift and, depending upon the degree of mixing, this drift could be less than or greater than that predicted by the linear solutions (20, 22). Note that the solutions (20) and (22) indicate now that a current meter would indeed measure a mean Eulerian current while a Lagrangian device would show no net displacement, in distinct contrast to the case in a flat bottom channel in the absence of rotation.

It may appear to be paradoxical that the mass and wave fields at an infinite distance are affected by the presence of the beach. But, this is just a result of the ellipticity of the full-non-linear system: the boundary values determine the mean interior fields. The linear wave equation is hyperbolic, but this character is an approximation.

An attempt is made in HOGG (1971) to compare these and more extensive calculations to observations near Bermuda. The difficulties, besides those of observing small average effects in the presence of large fluctuations, are that the solutions should depend partially upon the essentially unknown degree of mixing, and perhaps just as basically upon the fact that the observed internal wave field has a continuous spectrum and is not an infinitely long wave train at a single frequency. The calculations for a spectrum are considerably more difficult and are not attempted here. The calculation here has been greatly simplified, moreover, by restricting attention to the mean interactions; ultimately one would like to examine the possible slow modulations generated by non-harmonic interactions.

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